General Relativity

Solutions to HW 6

1. $G = 6.673 \times 10^{-11}\text{m}^3/(\text{kg s}^2)$, $c = 2.997924562 \times 10^8\text{m/s}$, $M_\odot = 1.989 \times 10^{30}\text{kg}$.
a) $[G] = L^3 M^{-1} T^{-2}$, $[c] = L/T$ Thus $[G/c^2] = M^{-1}L$, and hence $GM/c^2 = L$. So $\frac{G}{c^2}M_\odot = \frac{G}{c^2}1.989 \times 10^{30}\text{kg} = 1.476\text{km}$. In units where $G = c = 1$, $\frac{G}{c^2}M_\odot = M_\odot = 1.476\text{km}$.
b) $[GM/c^3] = T$. Thus $GM\omega/c^3$ is dimensionless. If $G = c = 1$, $M\omega$ is also dimensionless. So $GM_\odot\omega/c^3 = 0.00493$ or $M_\odot\omega = 0.00493$ if $G = c = 1$.
c) $[K] = (F/L^2)(M/L^3)^{-2} = ((ML/T^2)/L^2)(M/L^3)^{-2} = (L/T)^2(M/L^3)^{-1}$. Use $M_\odot$, $L_0 = \frac{G}{c^2}M_\odot$, and $T_0 = L_0/c = \frac{G}{c^3}M_\odot$ as units. Then $[K] = (L/T)^2 L^3/M) = (L_0/T_0)^2 L_0^3/M_\odot = c^2\frac{G^3}{c^6}M_\odot^3/M_\odot = c^{-4}G^3M_\odot^2$. Thus $KG^{-3}c^4M_\odot^{-2}$ is dimensionless. Here the latter has the value $KG^{-3}c^4M_\odot^{-2} = 123.6489$ and thus $K = 123.6489G^3c^{-4}M_\odot^2 = 0.01799\text{m}^5/(\text{kg s}^2)$.

2. a) Plugging in the definitions we get $-a_\mu V_\nu + \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}\Theta h_{\mu\nu} = ... = \nabla_\nu V_\mu$. Note in the Raychaudhuri eqn we use geodesics and thus $a = 0$.
b) Because $V_\mu V^\mu = -1$, $2V^\mu\nabla_\nu V_\mu = \nabla_\nu(V_\mu V^\mu) = 0$. Therefore

$$V^\mu a_\mu = V_\mu V^\nu \nabla_\nu V^\mu = 0$$

Putting this together gives

$$V^\mu \omega_{\mu\nu} = a_\nu/2 - V^\mu \nabla_\mu V_\nu/2 = 0.$$

Also, $V^\mu h_{\mu\nu} = 0$. Therefore,

$$V^\mu \sigma_{\mu\nu} = -a_\nu/2 + V^\mu \nabla_\mu V_\nu/2 - \Theta 0 = 0.$$

Finally,

$$\sigma_\mu{}^\mu = \nabla_\mu V^\mu - (\Theta/3)h_\mu{}^\mu = \nabla_\mu V^\mu - (\nabla_\mu V^\mu/3)3 = 0.$$

3. Consider a space like 4-vector S that is always perpendicular to the 4-velocity U of an accelerated particle, i.e. $U_\mu S^\mu = 0$.
a) In the local inertial frame $U^\mu = (1, 0, 0, 0)$, and thus $S^0 = 0$. So the vector S is purely spacelike in the local inertial frame: $S^\mu = (0, S^i)$.
b) Now we assume that the spatial components of S (which may be non-zero) are constant in the local inertial frame, i.e. $\frac{dS^i}{d\tau} = 0$. We also know that $U^i = 0$. Hence $\frac{dS^\mu}{d\tau} = kU^\mu$, where k is some scalar function.
c) Next we, take the proper time derivative of $U_\mu S^\mu = 0$ to find

$$0 = \frac{d}{d\tau}(U_\mu S^\mu) = \frac{dU_\mu}{d\tau}S^\mu + U_\mu \frac{dS^\mu}{d\tau} = \frac{dU_\mu}{d\tau}S^\mu + U_\mu k U^\mu = \frac{dU_\mu}{d\tau}S^\mu - k$$

and thus

$$k = \frac{dU^\mu}{d\tau}S_\mu.$$

d) Using b) and c) we find $\frac{dS^\mu}{d\tau} = kU^\mu = \frac{dU^\nu}{d\tau} S_\nu U^\mu$ in the local inertial frame. Replacing ordinary derivatives by covariant ones we thus have

$$\frac{DS^\mu}{d\tau} = \frac{DU^\nu}{d\tau} S_\nu U^\mu$$

valid in any frame. This is called Fermi-Walker transport of S. Note here we have $\frac{D}{d\tau} = U^\mu \nabla_\mu$.
e) Unlike e.g. the 3-momentum, the spin of a system in Newtonian physics is the same in all inertial frames. On the other hand, if spin is a four vector of SR it must transform as $S^{0'} = \gamma(S^0 - vS^x)$, $S^{x'} = \gamma(S^x - vS^0)$ under a boost in the x-direction. So for small v, we can have $S^{x'} \approx S^x$ only if $S^0 = 0$, i.e. $S^\mu = (0, S^i)$ must hold in the system's rest frame. We also know that the spin is constant in a local inertial frame if there is no torque. I.e. both $S^\mu = (0, S^i)$ and $\frac{dS^i}{d\tau} = 0$ must hold, which is what we assumed here. The equation in d) thus describes how the components of S^μ change if a particle with spin is accelerated (with 4-acceleration $a^\nu = \frac{DU^\nu}{d\tau}$) due to some external force, but without torques. Gravity also contributes to the spin change (via the Christoffel symbols in the covariant derivative). Without the external force $\frac{DU^\nu}{d\tau} = a^\nu = 0$, so that the RHS of d) is zero.
f) In the local inertial rest frame of the system: $S^\mu = (0, S^i)$, where $S^i = s^i$ is the spin 3-vector.
g) NOTE: $\frac{D}{d\tau}(S_\mu S^\mu) = 2S_\mu \frac{DS^\mu}{d\tau} = 2S_\mu \frac{DU^\nu}{d\tau} S_\nu U^\mu = 0$ since $S_\mu U^\mu = 0$.

Functional Derivative:

define $\dfrac{\delta}{\delta\Psi^B(y)}$ by: (Note: here $\Psi(x)$ is some field with indices A)

$$\frac{\delta\Psi^A(x)}{\delta\Psi^B(x')} := \delta^4(x-x')\,\delta^A_B \quad \& \quad \frac{\delta f(\Psi^A(x))}{\delta\Psi^B(x')} = f'(\Psi^A(x))\,\frac{\delta\Psi^A(x)}{\delta\Psi^B(x')}$$

$$\frac{\delta(\partial_\mu\Psi^A(x))}{\delta\Psi^B(x')} = \left[\frac{\partial}{\partial x^\mu}\delta^4(x-x')\right]\delta^A_B$$

ex: $S = -\frac{1}{2}\int(\partial^\mu\phi)\partial_\mu\phi\, d^4x$

$$\Rightarrow \frac{\delta S}{\delta\phi(x')} = -\frac{1}{2}\int 2(\partial^\mu\phi)\left(\frac{\delta}{\delta\phi(x')}\partial_\mu\phi(x)\right)d^4x = -\int(\partial^\mu\phi)\,\partial_\mu\,\delta^4(x-x')\,d^4x$$

$$= \int(\partial_\mu\partial^\mu\phi)\,\delta^4(x-x')\,d^4x = \partial_{\mu'}\partial^{\mu'}\phi(x')$$

EOM is $\dfrac{\delta S}{\delta\phi(x')} = 0 \iff \partial_\mu\partial^\mu\phi(x') = 0$

same result is obtained from:

$$\delta S = -\frac{1}{2}\int 2(\partial^\mu\phi)(\delta\,\partial_\mu\phi)\,d^4x = -\int(\partial^\mu\phi)\,\partial_\mu(\delta\phi)\,d^4x$$

$$= \int(\partial_\mu\partial^\mu\phi)\,\delta\phi\,d^4x$$

$$\delta S = 0 \Rightarrow \partial_\mu\partial^\mu\phi = 0$$

4.1 Carroll:

$\boxed{4.1}$ we know: $\dfrac{\delta\sqrt{-g(x')}}{\delta g^{\mu\nu}(x)} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta^4(x-x')$, $\dfrac{\delta g^{\alpha\mu}(x')}{\delta g^{\rho\sigma}(x)} = \delta^\alpha_\rho \delta^\mu_\sigma \delta^4(x-x')$

$\boxed{(a)}$

$S = \int \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A_\mu J^\mu \right] \sqrt{-g}\, d^4x'$ | Note: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $F^{\mu\nu} := g^{\alpha\mu} g^{\beta\nu} F_{\alpha\beta}$

consider case where $J^\mu = 0$ (no charges & currents)

$$\Rightarrow \frac{\delta S}{\delta g^{\rho\sigma}(x)} = \int \frac{\delta}{g^{\rho\sigma}(x)} \left(\left[-\frac{1}{4} g^{\alpha\mu} g^{\beta\nu} F_{\alpha\beta} F_{\mu\nu} \right] \sqrt{-g}\, d^4x' \right)$$

$$= -\frac{1}{4} \int \Big[\delta^\alpha_\rho \delta^\mu_\sigma g^{\beta\nu} F_{\alpha\beta} F_{\mu\nu}\, \delta^4(x'-x) \sqrt{-g} + g^{\alpha\mu} \delta^\beta_\rho \delta^\nu_\sigma F_{\alpha\beta} F_{\mu\nu}\, \delta^4(x'-x)\sqrt{-g}$$
$$+ g^{\alpha\mu} g^{\beta\nu} F_{\alpha\beta} F_{\mu\nu} \left(-\frac{1}{2}\sqrt{-g}\, g_{\rho\sigma}\, \delta^4(x'-x) \right) \Big] d^4x$$

$$= -\frac{1}{4} \left[g^{\beta\nu} F_{\rho\beta} F_{\sigma\nu} + g^{\alpha\mu} F_{\alpha\rho} F_{\mu\sigma} - \frac{1}{2} F^{\mu\nu} F_{\mu\nu}\, g_{\rho\sigma} \right] \sqrt{-g}$$

$$= -\frac{1}{2}\sqrt{-g} \left[F_{\rho\lambda} F_\sigma{}^\lambda - \frac{1}{4} g_{\rho\sigma} F^{\mu\nu} F_{\mu\nu} \right]$$

due to E&M alone

$$\Rightarrow \boxed{T_{\mu\nu}} := \frac{-2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \boxed{F_{\mu\lambda} F_\nu{}^\lambda - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}}$$

if we include $A_\mu J^\mu$ terms, we also need to include matter action

$\boxed{(b)}$ consider $S' = \int \beta R^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} \sqrt{-g}\, d^4x$

~~recall $\frac{\delta R^{\alpha\beta}(x')}{\delta g^{\mu\nu}(x)}$~~ with $F_{\mu\rho} = \partial_\mu A_\rho - \partial_\rho A_\mu$

or $\mathcal{L}' = \beta R^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}$

use Euler – Lagrange: $\dfrac{\partial \mathcal{L}}{\partial A_\nu} = \nabla_\mu \left(\dfrac{\partial \mathcal{L}}{\partial(\nabla_\mu A_\nu)} \right)$

Here $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu + \mathcal{L}'$, $\quad \mathcal{L}' = \beta R^{\alpha\gamma} g^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta}$

$\Rightarrow \dfrac{\partial \mathcal{L}}{\partial A_\mu} = J^\mu \leftarrow$ as before

to obtain $\dfrac{\partial \mathcal{L}}{\partial(\nabla_\mu A_\nu)}$ note: $\dfrac{\partial F_{\alpha\beta}}{\partial(\nabla_\mu A_\nu)} = \delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha$

$$\Rightarrow \frac{\partial F_{\alpha\beta} F^{\alpha\beta}}{\partial(\nabla_\mu A_\nu)} = 4 F_{\mu\nu} \Rightarrow \frac{\partial \mathcal{L}}{\partial(\nabla_\mu A_\nu)} = -F^{\mu\nu} + \frac{\partial \mathcal{L}'}{\partial(\nabla_\mu A_\nu)}$$

$$\frac{\partial \mathcal{L}'}{\partial(\nabla_\mu A_\nu)} = \beta R^{\alpha\gamma} g^{\beta\delta} \left[(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha) F_{\gamma\delta} + F_{\alpha\beta} (\delta^\mu_\gamma \delta^\nu_\delta - \delta^\mu_\delta \delta^\nu_\gamma) \right]$$

$$= \beta \left[(R^{\mu\gamma} g^{\nu\delta} - R^{\nu\gamma} g^{\mu\delta}) F_{\gamma\delta} + (R^{\alpha\mu} g^{\beta\nu} - R^{\alpha\nu} g^{\beta\mu}) F_{\alpha\beta} \right]$$

$$= \beta \left[2 (R^\mu{}_\gamma F^{\gamma\nu} - R^\nu{}_\gamma F^{\gamma\mu}) \right] \quad \Big| \quad R^\mu{}_\gamma = G^\mu{}_\gamma + \tfrac{1}{2} R g^\mu{}_\gamma$$

$$= 2\beta \left[F^{\gamma\nu} G^\mu{}_\gamma - F^{\gamma\mu} G^\nu{}_\gamma + \tfrac{1}{2} R F^{\mu\nu} - \tfrac{1}{2} R F^{\nu\mu} \right]$$

$$= 2\beta \left[F^{\gamma\nu} G^\mu{}_\gamma - F^{\gamma\mu} G^\nu{}_\gamma + R F^{\mu\nu} \right]$$

$$\nabla_\mu \left(\frac{\partial \mathcal{L}'}{\partial(\nabla_\mu A_\nu)} \right) = 2\beta \left[(\nabla_\mu F^{\gamma\nu}) G^\mu{}_\gamma - (\nabla_\mu F^{\gamma\mu}) G^\nu{}_\gamma - F^{\gamma\mu} \nabla_\mu G^\nu{}_\gamma + (\nabla_\mu R) F^{\mu\nu} + R \nabla_\mu F^{\mu\nu} \right]$$

So EOM is:

$$\boxed{-\nabla_\mu F^{\mu\nu} + 2\beta \left[(\nabla_\mu F^{\gamma\nu}) G^\mu{}_\gamma - (\nabla_\mu F^{\gamma\mu}) G^\nu{}_\gamma - F^{\gamma\mu} \nabla_\mu G^\nu{}_\gamma + (\nabla_\mu R) F^{\mu\nu} + R \nabla_\mu F^{\mu\nu} \right] = J^\nu}$$

with $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$

What is $\nabla_\mu \mathcal{J}^\mu$?

$$\nabla_\mu \mathcal{J}^\mu = \nabla_\nu \mathcal{J}^\nu = -\nabla_\nu \nabla_\mu F^{\mu\nu} + 2\beta \nabla_\nu \nabla_\mu \left[F^{\gamma\nu} G^\mu{}_\gamma - F^{\gamma\mu} G^\nu{}_\gamma + R F^{\mu\nu} \right]$$

$$= 0 + 2\beta \left[\nabla_\mu \nabla_\nu \left(F^{\gamma\mu} G^\nu{}_\gamma \right) - \nabla_\nu \nabla_\mu \left(F^{\gamma\mu} G^\nu{}_\gamma \right) + 0 \right]$$

$$= 2\beta \left[\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu \right] \left(F^{\gamma\mu} G^\nu{}_\gamma \right)$$

Note: $G_{\nu\gamma} = 8\pi T_{\nu\gamma}$

so we seem to need the new $T_{\nu\gamma}$ if we want to know if $\nabla_\mu \mathcal{J}^\mu$ is zero.

$$T_{\nu\gamma} = \frac{-2}{\sqrt{-g}} \frac{\delta}{\delta g^{\nu\gamma}} \int \mathcal{L} \sqrt{-g}\, d^4x$$ ←) leave this as an exercise

But $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) C_{\alpha\beta} = R_{\mu\nu\alpha}{}^\lambda C_{\lambda\beta} + R_{\mu\nu\beta}{}^\lambda C_{\alpha\lambda}$

$$\Rightarrow \underline{(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) C^{\mu\nu}} = \underbrace{R_{\mu\nu}{}^{\mu\lambda}}_{R_\nu{}^\lambda} C_\lambda{}^\nu + \underbrace{R_{\mu\nu}{}^{\nu\lambda}}_{-R_\mu{}^\lambda} C^\mu{}_\lambda = R_{\nu\lambda} C^{\lambda\nu} - R_{\mu\lambda} C^{\mu\lambda}$$

$$= R_{\alpha\beta} C^{\beta\alpha} - R_{\beta\alpha} C^{\beta\alpha} = \underline{0}$$

here $C^{\mu\nu} = F^{\gamma\mu} G^\nu{}_\gamma$

$$\Rightarrow \underline{\nabla_\mu \mathcal{J}^\mu = 0}$$

4.3 Carroll

$\boxed{4.3}$ Note: $\int \delta(y-x) f(x)\,dx = f(y)$

$$\Rightarrow \frac{\partial}{\partial y}\int \delta(y-x) f(x)\,dx = \frac{\partial}{\partial y} f(y)$$

$$\int \left(\frac{\partial}{\partial y}\delta(y-x)\right) f(x)\,dx = \partial_y f(y)$$

$$\int \left(-\frac{\partial}{\partial x}\delta(y-x)\right) f(x)\,dx = \int \delta(y-x)\frac{\partial}{\partial x} f(x)\,dx = \partial_y f(y)$$

$$\Rightarrow \left(\frac{\partial}{\partial y}\delta(y-x)\right) = \left(-\frac{\partial}{\partial x}\delta(y-x)\right) = \delta(y-x)\frac{\partial}{\partial x}$$

in GR: $\displaystyle\int \frac{\delta^4(y-x)}{\sqrt{-g}} f(x)\sqrt{-g}\,d^4x = f(y)$

for dust $T^{\mu\nu} = \rho U^\mu U^\nu$

consider single particle : $U^\mu = \dfrac{dx^\mu}{d\tau}$, $\cancel{\rho = m\frac{\delta^4(y-x)}{\sqrt{-g}}}$

$\Downarrow$

$$\boxed{T^{\mu\nu}(y) = \int \frac{m\,\delta^4(y-x(\tau))}{\sqrt{-g}}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,d\tau} \quad (*)$$

~~def: $x'^\mu = y^\mu$~~

$$\cancel{\Rightarrow T^{\mu\nu}(x') = m\int \frac{\delta^4(x'-x)}{\sqrt{-g}}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,d\tau}$$

Note: $T^{\mu\nu}(y)$ depends only on y not on x.

let $\nabla_\mu(y) := \dfrac{\partial}{\partial y^\mu}$ in local inert. frame

$\nabla_\mu := \dfrac{\partial}{\partial x^\mu}$ " " " "

$\Downarrow$

$$\underbrace{\nabla_\mu(y)\,T^{\mu\nu}(y)}_{=0} = m\int \left[\nabla_\mu(y)\frac{\delta^4(y-x(\tau))}{\sqrt{-g}}\right]\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,d\tau$$

4.3 Carroll (simplified solution in local inertial frame):

go to local inertial frame at point y

$$\Rightarrow U^\mu = \frac{dx^\mu}{d\tau} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \Leftrightarrow x^0 = \tau \quad , x^i = 0$$

$$\Rightarrow 0 = m \int \left(\frac{\partial}{\partial y^\mu} \delta^4(y - x(\tau))\right) \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\, d\tau \qquad \Big| \; \frac{dx^i}{d\tau} = 0$$

$$= m \int \left(\frac{\partial}{\partial y^0} \delta^4(y - x(\tau))\right) \frac{dx^0}{d\tau}\frac{dx^\nu}{d\tau}\, d\tau$$

$$\cancel{= m \int \delta^4(y - x(\tau)) \frac{\partial}{\partial x^0}}$$

$$= m\, \delta^3(y^i - x^i) \int \left[\frac{\partial}{\partial y^0} \delta(y^0 - x^0)\right] \frac{dx^0}{d\tau}\frac{dx^\nu}{d\tau}\, d\tau$$

$$= m\, \delta^3(y^i - x^i) \int \delta(y^0 - x^0) \frac{\partial}{\partial x^0}\left[\frac{dx^0}{d\tau}\frac{dx^\nu}{d\tau}\right] d\tau \qquad \Big| \; x^0 = \tau$$

$$= m \int \delta^4(y - x(\tau)) \left[\frac{d^2x^0}{d\tau^2}\frac{dx^\nu}{d\tau} + \frac{dx^0}{d\tau}\frac{d^2x^\nu}{d\tau^2}\right] d\tau$$

in order for this to be zero the square bracket has to be zero!

$$\text{if } \nu = i \Rightarrow \left[\frac{d^2x^0}{d\tau^2} \cdot 0 + 1 \cdot \frac{d^2x^i}{d\tau^2}\right] = 0$$

$$\text{if } \nu = 0 \Rightarrow \left[\frac{d^2x^0}{d\tau^2} \cdot 1 + 1 \cdot \frac{d^2x^0}{d\tau^2}\right] = 0$$

$$\Rightarrow \boxed{\frac{d^2x^\nu}{d\tau^2} = 0}$$

in any frame: $\boxed{\frac{D}{d\tau}\frac{dx^\nu}{d\tau} = 0} \Leftrightarrow \boxed{U^\mu \nabla_\mu U^\nu = 0}$

geodesic eqn, where we use an affine par

Note about 4.3 in Carroll:
Carroll starts with dust, that has

$$T^{\mu\nu} = \rho U^\mu U^\nu$$

where ρ is the energy density and U^μ the 4-velocity of dust.
If we use $T^{\mu\nu} = \rho U^\mu U^\nu$ in $0 = \nabla_\mu T^{\mu\nu}$ we find

$$0 = \nabla_\mu T^{\mu\nu} = \nabla_\mu(\rho U^\mu U^\nu) = [\nabla_\mu(\rho U^\mu)]U^\nu + \rho U^\mu \nabla_\mu U^\nu$$

Contracting with U_ν then yields

$$0 = [\nabla_\mu(\rho U^\mu)]U^\nu U_\nu + \rho U^\mu U_\nu \nabla_\mu U^\nu.$$

Of course $U^\nu U_\nu = -1$ and thus $U_\nu \nabla_\mu U^\nu = 0$, and hence

$$0 = -\nabla_\mu(\rho U^\mu)$$

If we insert this above, $0 = \nabla_\mu T^{\mu\nu}$ becomes

$$0 = \nabla_\mu T^{\mu\nu} = \rho U^\mu \nabla_\mu U^\nu$$

and thus U^ν satifies the geodesic equation. Hence the 4-velocity of dust obeys the geodesic equation. But this is not really what the problem asked for. In the problem we are supposed to use the special $T^{\mu\nu}$ given in terms of the δ-function.

4.3 Carroll (real solution):
Here we use a, b, c, ... instead of greek letters for the components, because this was easier to type in latex. But we do not use abstract indices.
We start with

$$T^{ab}(y) = m \int \frac{\delta^4(y - x(\tau))}{\sqrt{|g|}} \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} d\tau$$

We need to use $\nabla_a(y)T^{ab}(y) = 0$. We can write $\nabla_a T^{ab} = \partial_a T^{ab} + \Gamma^a_{ac} T^{cb} + \Gamma^b_{ac} T^{ac}$. Next we use $\Gamma^a_{ac} = \frac{1}{\sqrt{|g|}} \partial_c \sqrt{|g|}$. Thus

$$\nabla_a T^{ab} = \frac{1}{\sqrt{|g|}} \partial_a(\sqrt{|g|} T^{ab}) + \Gamma^b_{ac} T^{ac}. \tag{1}$$

Concentrate on
$\partial_a(y)(\sqrt{|g|}T^{ab}(y)) = m \int \partial_a(y)\delta^4(y - x(\tau)) \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} d\tau = m \int [-\partial_a(x)\delta^4(y - x(\tau))] \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} d\tau$
$= -m \int [\frac{dx^a}{d\tau} \partial_a(x)\delta^4(y - x(\tau))] \frac{dx^b}{d\tau} d\tau = -m \int [\frac{d}{d\tau}\delta^4(y - x(\tau))] \frac{dx^b}{d\tau} d\tau$
$= -m \int \frac{d}{d\tau}[\delta^4(y - x(\tau)) \frac{dx^b}{d\tau}] d\tau + m \int \delta^4(y - x(\tau)) \frac{d}{d\tau} \frac{dx^b}{d\tau} d\tau$
$= -m \left[\delta^4(y - x(\tau)) \frac{dx^b}{d\tau}\right]_{\tau=-\infty}^{\tau=\infty} + m \int \delta^4(y - x(\tau)) \frac{d^2 x^b}{d\tau^2} d\tau.$
The first term vanishes, because $\lim_{\tau\to\pm\infty} x^0(\tau) = \pm\infty$ and thus $\lim_{\tau\to\pm\infty} \delta(y^0 - x^0(\tau)) = 0$ for any finite y^0. So that:

$$\partial_a(y)(\sqrt{|g|}T^{ab}(y)) = m \int \delta^4(y - x(\tau)) \frac{d^2 x^b}{d\tau^2} d\tau$$

Thus Eq. (1) yields

$$\nabla_a(y)T^{ab}(y) = m\int \frac{\delta^4(y-x(\tau))}{\sqrt{|g|}}\frac{d^2x^b}{d\tau^2}d\tau + \Gamma^b_{ac}T^{ac} = m\int \frac{\delta^4(y-x(\tau))}{\sqrt{|g|}}\left[\frac{d^2x^b}{d\tau^2} + \Gamma^b_{ac}\frac{dx^a}{d\tau}\frac{dx^c}{d\tau}\right]d\tau$$

In order for this to be zero the square brackt has to be zero, i.e. the geodesic eqn

$$\frac{d^2x^b}{d\tau^2} + \Gamma^b_{dc}\frac{dx^d}{d\tau}\frac{dx^c}{d\tau} = 0$$

has to hold.
Recall:
$\int \delta(y-vt)f(t)dt = \int \delta(y-s)f(s/v)ds/v = f(y/v)/v$ and thus $\delta(y-x(t)) = \delta(y/\dot{x}-t)/\dot{x}$.

4.3 Carroll (alternative real solution, using a test function):
We start with

$$T^{ab}(y) = m\int d\tau \frac{\delta^4(y-x(\tau))}{\sqrt{|g|}}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau}$$

We need to use $\nabla_a(y)T^{ab}(y) = 0$. We can write

$$I := \int d^4y\sqrt{|g|}f_b(y)\nabla_a(y)T^{ab}(y) = 0,$$

where $f_b(y)$ is a test function that we choose to have compact support, i.e. $\lim_{y^c\to\pm\infty} f_b(y) = 0$. Then $I = \int d^4y\sqrt{|g|}f_b(y)\nabla_a(y)T^{ab}(y) = \int d^4y\sqrt{|g|}(\nabla_a(y)[f_b(y)T^{ab}(y)] - T^{ab}(y)\nabla_a(y)f_b(y)) = \oint dAn^a f_b T^{ab} - \int d^4y\sqrt{|g|}T^{ab}(y)\nabla_a(y)f_b(y) = -\int d^4y\sqrt{|g|}T^{ab}(y)\nabla_a(y)f_b(y)$, where the surface integral was dropped because $\lim_{y^c\to\pm\infty} f_b(y) = 0$.
Then $I = -m\int d\tau\int d^4y\sqrt{|g|}\frac{\delta^4(y-x(\tau))}{\sqrt{|g|}}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau}\nabla_a(y)f_b(y) = -m\int d\tau\frac{dx^a}{d\tau}\frac{dx^b}{d\tau}\nabla_a f_b = -m\int d\tau U^b\frac{Df_b}{d\tau}$, where we have used $U^a = \frac{dx^a}{d\tau}$ and $U^a\nabla_a f_b = \frac{Df_b}{d\tau}$.
Next note that $\frac{D}{d\tau}(U^b f_b) = U^b\frac{Df_b}{d\tau} + \frac{DU^b}{d\tau}f_b$, so that $I = m\int d\tau[\frac{DU^b}{d\tau}f_b - \frac{D}{d\tau}(U^b f_b)] = m\int d\tau\frac{DU^b}{d\tau}f_b - m\left[U^b f_b\right]^{\tau=\infty}_{\tau=-\infty}$. Since $\lim_{\tau\to\pm\infty} x^0(\tau) = \pm\infty$, we have $\lim_{\tau\to\pm\infty} f_b(x(\tau)) = 0$, for all f_b with compact support. Thus $0 = I = m\int d\tau\frac{DU^b}{d\tau}f_b$ holds for all f_b with compact support. This can only be true if

$$\frac{DU^b}{d\tau} = 0,$$

i.e. for a geodesic.

4.6 Carroll:
The Maxwell Eqs $\nabla_\mu F^{\nu\mu} = J^\nu$ and $\nabla_{[\rho}F_{\nu\mu]} = 0$ can be written in terms of A_ν: Let $F = dA$ (i.e. $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$) then we get

$$\nabla_\mu\nabla^\nu A^\mu - \nabla_\mu\nabla^\mu A^\nu = J^\nu.$$

Assume vacuum ($J^\nu = 0$) and $A_\mu = K_\mu$ with $\nabla_\nu K_\mu = -\nabla_\mu K_\nu$, then we get

$$-2\nabla_\mu\nabla^\mu K^\nu = 0.$$

Considering Eq. (3.176) $\nabla_\mu\nabla_\nu K_\lambda = R_{\lambda\nu\mu\rho}K^\rho$ we find $g^{\mu\nu}\nabla_\mu\nabla_\nu K_\lambda = -R_{\lambda\rho}K^\rho$. The latter is zero in vacuum and thus the Maxwell Eqn is satisfied.