General Relativity

Solutions to HW 3

1. Particle collision

$$A + B \longrightarrow C + D + E$$

$$p_{(A)} = (3,-1,0,0)\ ,\quad p_{(B)} = (2,1,1,0)\ ,\quad p_{(C)} = (1,1,0,0)\ ,\quad p_{(D)} = (1,-\tfrac{1}{2},0,0)$$

mom. cons: $p_{(A)} + p_{(B)} = p_{(C)} + p_{(D)} + p_{(F)}$

$$\Rightarrow p^{\alpha}_{(F)} = (3,-\tfrac{1}{2},1,0) \quad \Rightarrow p^{\alpha}_{(F)} p^{\beta}_{(F)} \eta_{\alpha\beta} = \frac{-31}{4}$$

$\Rightarrow$ rest mass of F is $m_F = \frac{1}{2}\sqrt{31}$ $\quad \left(m_F = \sqrt{-p^{\alpha}_{(F)} p_{(F)\alpha}} \right)$

CM has mom: $p_{cm} = p_{(A)} + p_{(B)} = (5,0,1,0)$

$$\Rightarrow p_{cm} = M_{cm} u_{cm}\ ,\quad M_{cm} = \sqrt{-p^{\alpha}_{cm} p_{cm\,\alpha}} = \sqrt{24}$$

$$\Rightarrow u_{cm} = \frac{p_{cm}}{M_{cm}} = \frac{5}{\sqrt{24}}(1,0,\tfrac{1}{5},0) \overset{!}{=} \gamma(1, V_{cm})$$

$$\Rightarrow \gamma = \frac{5}{\sqrt{24}} \quad \& \quad V_{cm} = (0,\tfrac{1}{5},0)$$

2. Projection tensor

U is timelike unit 4 vector, (e.g. 4-vel.)

$\underline{U_\alpha U^\alpha = -1}$

$\underline{P := g + U \otimes U} \iff P_{\alpha\beta} = g_{\alpha\beta} + U_\alpha U_\beta$

$\underline{V_\perp^\alpha := P^\alpha{}_\beta V^\beta}$

(a) $\underline{U_\alpha V_\perp^\alpha} = U_\alpha (g^\alpha{}_\beta + U^\alpha U_\beta) V^\beta = U_\alpha V^\alpha + \overbrace{(U_\alpha U^\alpha)}^{=-1} U_\beta V^\beta$

$\underline{= 0}$

(b) $\underline{P^\alpha{}_\beta V_\perp^\beta} = (g^\alpha{}_\beta + U^\alpha U_\beta)(g^\beta{}_\gamma + U^\beta U_\gamma) V^\gamma$

$= (g^\alpha{}_\gamma + U^\alpha U_\gamma \cdot 2 + U^\alpha U_\gamma \underbrace{U_\beta U^\beta}_{=-1}) V^\gamma = P^\alpha_\gamma V^\gamma = \underline{V_\perp^\alpha}$

c) $P_{\alpha\beta} P^\alpha{}_\mu P^\beta{}_\nu V^\mu W^\nu - g_{\alpha\beta} P^\alpha{}_\mu P^\beta{}_\nu V^\mu W^\nu$

$= S_{\mu\nu} V^\mu W^\nu$ where $S_{\mu\nu} := (P_{\alpha\beta} - g_{\alpha\beta}) P^\alpha{}_\mu P^\beta{}_\nu$

note: $\left.\begin{array}{l} P_{\alpha\beta} P^\alpha{}_\mu = P_{\beta\mu} \Leftarrow \text{easy to see} \\ g_{\alpha\beta} P^\alpha{}_\mu = P_{\beta\mu} \end{array}\right\} \Rightarrow S_{\mu\nu} = 0$

$\Rightarrow \underline{P(V_\perp, W_\perp) = g(V_\perp, W_\perp)}$

if U is not a unit vector

$$P_{\alpha\beta} = g_{\alpha\beta} - \frac{U_\alpha U_\beta}{U_\lambda U^\lambda}$$

3. Problems in Carroll:

2.4, Carroll

$$[X,Y](f) \equiv X(Y(f)) - Y(X(f)). \tag{1}$$

Linearity:

$$[X,Y](af+bg) = X(Y(af+bg)) - Y(X(af+bg)). \tag{2}$$

X and Y are vector fields and hence obey linearity etc., giving

$$\begin{aligned}
[X,Y](af+bg) &= X(aY(f)+bY(g)) - Y(aX(f)+bX(g)) &(3)\\
&= aX(Y(f)) + bX(Y(g)) - aY(X(f)) - bY(X(g)) &(4)\\
&= a[X,Y](f) + b[X,Y](g). &(5)
\end{aligned}$$

Leibniz:

$$\begin{aligned}
[X,Y](fg) &= X(Y(fg)) - Y(X(fg)) &(6)\\
&= X(fY(g)+Y(f)g) - Y(fX(g)+X(f)g) &(7)\\
&= fX(Y(g)) + X(f)Y(g) + X(Y(f))g + Y(f)X(g) - &(8)\\
&\qquad -fY(X(g) - Y(f)(X(g) - Y(X(f))g - X(f)Y(g) &(9)\\
&= f[X,Y](g) + ([X,Y](g))f. &(10)
\end{aligned}$$

Component formula:

$$\begin{aligned}
[X,Y] &= [X^\mu\partial_\mu, Y^\nu\partial_\nu] &(11)\\
&= X^\mu\partial_\mu Y^\nu\partial_\nu + X^\mu Y^\nu\partial_\mu\partial_\nu - Y^\nu\partial_\nu X^\mu\partial_\mu - Y^\nu X^\mu\partial_\nu\partial_\mu &(12)\\
&= (X^\mu\partial_\mu Y^\nu - Y^\mu\partial_\mu X^\nu)\partial_\nu. &(13)
\end{aligned}$$

Tranformation as a vector field:

$$\begin{aligned}
[X,Y]^\nu &= X^\mu\partial_\mu Y^\nu - Y^\mu\partial_\mu X^\nu &(14)\\
&= \frac{\partial x^\mu}{\partial x^{\rho'}}X^{\rho'}\frac{\partial x^{\lambda'}}{\partial x^\mu}\partial_{\lambda'}\frac{\partial x^\nu}{\partial x^{\sigma'}}Y^{\sigma'} - \frac{\partial x^\mu}{\partial x^{\rho'}}Y^{\rho'}\frac{\partial x^{\lambda'}}{\partial x^\mu}\partial_{\lambda'}\frac{\partial x^\nu}{\partial x^{\sigma'}}X^{\sigma'} &(15)\\
&= \frac{\partial x^\nu}{\partial x^{\sigma'}}X^{\rho'}\partial_{\rho'}Y^{\sigma'} - \frac{\partial x^\nu}{\partial x^{\sigma'}}Y^{\rho'}\partial_{\rho'}X^{\sigma'} &(16)\\
&= \frac{\partial x^\nu}{\partial x^{\sigma'}}[X,Y]^{\sigma'}. &(17)
\end{aligned}$$

2.5, Carroll Any examples which work will do here, i.e., any two vector fields which are non-zero everywhere and which do not commute will do. A simple example is $(y, 1)$ and $(1, x)$.

2.6 Carroll:

$$x(\lambda) = \cos(\lambda), \qquad y(\lambda) = \sin(\lambda), \qquad z(\lambda) = \lambda$$

In polar coordinates,

$$r(\lambda) = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + \lambda^2}$$

$$\theta(\lambda) = \arctan \sqrt{\frac{x^2 + y^2}{z^2}} = \arctan \frac{1}{\lambda}$$

$$\phi(\lambda) = \arctan \frac{y}{x} = \lambda$$

Tangent vectors are computed from $\frac{dx^\mu}{d\lambda}$. In cartesian coordinates the tangent vector is $(-\sin(\lambda), \cos(\lambda), 1)$ and in spherical coordinates it is $(\frac{\lambda}{\sqrt{1+\lambda^2}}, \frac{-1}{1+\lambda^2}, 1)$.

2.8 Carroll: Exterior derivative of wedge product

η is a q-form, ω a p-form

$$(d(\omega\wedge\eta))_{\mu_1\cdots\mu_{p+1+q}} \overset{\text{use defs of } d\,\&\,\wedge}{=} (p+1+q)\,\partial_{[\mu_1} \frac{(p+q)!}{p!\,q!}\,\omega_{\mu_2\cdots\mu_{p+1}}\eta_{\mu_{p+2}\cdots\mu_{p+1+q}]}$$

$$= \frac{(p+1+q)!}{p!\,q!}\,\partial_{[\mu_1}\omega_{\mu_2\cdots\mu_{p+1}}\eta_{\mu_{p+2}\cdots\mu_{p+1+q}]}$$

$$= \frac{(p+1+q)!}{p!\,q!}\Big\{\big(\partial_{[\mu_1}\omega_{\mu_2\cdots\mu_{p+1}}\big)\eta_{\mu_{p+2}\cdots\mu_{p+1+q}]} + \omega_{[\mu_2\cdots\omega_{p+1}}\,\partial_{\mu_1}\eta_{\mu_{p+2}\cdots\mu_{p+1+q}]}\cdot(-1)^p\Big\}$$

for each of μ_1 with the ind $\mu_2\cdots\mu_{p+1}$

$$= \frac{(p+1+q)!}{(p+1)!\,q!}\,(p+1)\big(\partial_{[[\mu_1}\omega_{\mu_2\cdots\mu_{p+1}]}\big)\eta_{\mu_{p+2}\cdots\mu_{p+1+q}]}$$

$$+ \frac{(p+1+q)!}{p!\,(q+1)!}\,\omega_{[\mu_2\cdots\mu_{p+1}}\,\partial_{[\mu_1}\eta_{\mu_{p+2}\cdots\mu_{p+1+q}]]}\cdot(q+1)\cdot(-1)^p$$

$$\Rightarrow \underline{(d\omega)\wedge\eta + (-1)^p\,\omega\wedge(d\eta) = d(\omega\wedge\eta)}$$

Note: here we used

$$A_{[\mu_1\mu_2[\mu_3\cdots\mu_n]]} = A_{[\mu_1\cdots\mu_n]}$$