General Relativity - PHY 6938

HW 6

Hand in this homework.

READ: Chap 4.1-4.8, Appendix F up to Eq. (F.11)

PROBLEMS:

1. This problem is about units:
a) Assuming $G = c = 1$, express the mass of the sun $M_\odot = 1.989 \times 10^{30}$kg in km.
b) Consider an angular frequency of $\omega = 1000/\text{s}$. Show that $M\omega$ is dimensionless if we use $G = c = 1$. What is the value of $M_\odot\omega$ in units of $G = c = 1$.
c) A particular polytropic equation of state has the form $p = K\rho_0^2$, where p is the pressure, ρ_0 the rest mass density and K a constant. Assume K has the value $K = 123.6489$ in units where $G = c = M_\odot = 1$. What is the value of K in SI units?

2. Let V be a timelike unit vector field V (so $g_{\mu\nu}V^\mu V^\nu = -1$).
a) Show that its covariant derivative can be decomposed as

$$\nabla_\nu V_\mu = -a_\mu V_\nu + \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}\Theta h_{\mu\nu}$$

where $\quad a^\mu = V^\nu\nabla_\nu V^\mu, \quad \Theta = \nabla_\nu V^\nu, \quad h_{\mu\nu} = g_{\mu\nu} + V_\mu V_\nu$

$$\omega_{\mu\nu} = \frac{1}{2}(a_\mu V_\nu - a_\nu V_\mu) + \frac{1}{2}(\nabla_\nu V_\mu - \nabla_\mu V_\nu), \quad \sigma_{\mu\nu} = \frac{1}{2}(a_\mu V_\nu + a_\nu V_\mu) + \frac{1}{2}(\nabla_\nu V_\mu + \nabla_\mu V_\nu) - \frac{1}{3}\Theta h_{\mu\nu}.$$

This is used in the derivation of the Raychaudhuri equation.
b) Compute $V^\mu\omega_{\mu\nu}$, $V^\mu\sigma_{\mu\nu}$, $V^\mu h_{\mu\nu}$, $V^\mu a_\mu$ and $g^{\mu\nu}\sigma_{\mu\nu}$.

3. Consider a spacelike 4-vector S that is always perpendicular to the 4-velocity U of an accelerated system, i.e. $U_\mu S^\mu = 0$.
a) What does this imply for the components of S in the local inertial rest frame of the system?
b) Assume that $\frac{dS^i}{d\tau} = 0$ in the local inertial rest frame of the system ($i = 1, 2, 3$). Show that then $\frac{dS^\mu}{d\tau} = kU^\mu$ must hold in the local inertial rest frame.
c) Find the function k.
d) Find an expression for $\frac{DS^\mu}{d\tau}$ that is valid in any frame.
e) Convince yourself that S can describe the spin of a system on which no torque acts, and that the equation found in d) describes how the spin components change due to an external force and gravity, when the torque is zero. The expression found in d) is called Fermi-Walker transport of S.
f) Express all components of S in terms of the spin 3-vector s^i in the local inertial rest frame of the system.

4. Do the following problems from Carroll's book:
4.1 (in part (b) find only new EOM for A_μ, i.e. you do not have to answer the questions

about the Einstein eqn and the current),
4.3,
4.6 (Hint: use 10.7 from problem book)