HW 9

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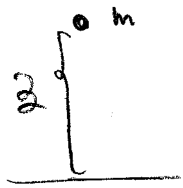

at time t, the momentum of the particle is $p(t)$ and the height is z.

$$H = \frac{p^2}{2m} + mgz = E \leftarrow \text{Total energy}$$

since $\frac{\partial H}{\partial t} = 0 \Rightarrow S = W - \alpha t$

$$\alpha = E = \text{const.}$$

$$p = \frac{\partial S}{\partial z} = \frac{\partial W}{\partial z}, \quad \frac{\partial S}{\partial t} = -\alpha = -E$$

$$\frac{1}{2m}\left(\frac{\partial W}{\partial z}\right)^2 + mgz = \alpha$$

$$\frac{\partial W}{\partial z} = \sqrt{2m(\alpha - mgz)}$$

$$W = (2m)^{1/2}\int (\alpha - mgz)^{1/2}\,dz$$

$$= -\frac{2}{3}\frac{\sqrt{2m}}{mg}(\alpha - mgz)^{3/2}$$

$$S(z,t) = -\frac{2}{3g}\left(\frac{2}{m}\right)^{1/2}(\alpha - mgz)^{3/2} - \alpha t$$

where $\alpha = E$

$$\beta' = \frac{\partial S}{\partial \alpha}$$

$$= -\frac{\cancel{2}}{\cancel{3}g}\left(\frac{2}{m}\right)^{1/2} \frac{\cancel{3}}{\cancel{2}} (\alpha - mgz)^{1/2} \cdot 1 - t$$

$$\beta' + t = -\frac{1}{g}\sqrt{\frac{2(\alpha - mgz)}{m}} \qquad \text{——— } (1)$$

$$mg^2(\beta' + t)^2 = 2(\alpha - mgz)$$

$$z = \frac{\cancel{2}\alpha}{\cancel{2}mg} - \frac{\cancel{m}g^{\cancel{2}}(\beta' + t)^2}{2\cancel{m}\cancel{g}}$$

$$z = \frac{\alpha}{mg} - \frac{1}{2}g(\beta' + t)^2 \qquad \text{——— } (2)$$

$$p = \frac{\partial W}{\partial z} = \sqrt{2m(\alpha - mgz)}$$

$$p \cdot \left(-\frac{1}{gm}\right) = -\frac{1}{g}\sqrt{\frac{2m(\alpha - mgz)}{m}} = \beta' + t \quad (\text{from } (1))$$

$$p = -mg(\beta' + t)$$

at $t = 0$ $\quad p(0) = p_0 \quad \Rightarrow \quad p(0) = p_0 = -mg\beta'$

$$\beta' = \frac{-p_0}{mg}$$

$$p(t) = -mg\left(\frac{-p_0}{mg} + t\right)$$

$$p(t) = \underline{p_0 - mgt}$$

2 $\Rightarrow t=0$

$$z(0) = z_0 = \frac{\alpha}{mg} - \frac{1}{2}\cancel{g}\left(\frac{p_0^2}{m^2 g \cancel{x}}\right)$$

$$z_0 = \frac{\alpha}{mg} - \frac{p_0^2}{2gm^2}$$

$$z(t) = \underbrace{\frac{\alpha}{mg} - \frac{1}{2}g\beta'^2}_{} - \frac{1}{2}gt^2 - g\beta' t$$

$$= z_0 - \cancel{g}t\left(\frac{-p_0}{m\cancel{g}}\right) - \frac{1}{2}gt^2$$

$$z(t) = z_0 + \frac{p_0}{m} - \frac{1}{2}gt^2 \checkmark$$

$$p(t) = p_0 - mgt \checkmark$$

|2|

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + mgy = E \leftarrow \text{energy}$$

Since $\dfrac{\partial H}{\partial t} = 0 \Rightarrow S = W - \alpha t$ ✓

x is cyclic $\Rightarrow W = W_y + \alpha_x x$ ✓

$\uparrow$ constant

$E = \alpha$

H–J

$$\frac{1}{2m}\left[\alpha_x^2 + \left(\frac{\partial W_y}{\partial y}\right)^2\right] + mgy = \alpha$$

$$\left(\frac{\partial W_y}{\partial y}\right)^2 + \alpha_x^2 = 2m\alpha - 2m^2 gy$$

$$\frac{\partial W_y}{\partial y} = \sqrt{2m\alpha - \alpha_x^2 - 2m^2 g y}$$

$$W_y = \int \left[2m\alpha - \alpha_x^2 - 2m^2 g y\right]^{\frac{1}{2}} dy$$

$$W_y = -\frac{\cancel{2}}{3}\frac{1}{\cancel{2}m^2 g}\left[2m\alpha - \alpha_x^2 - 2m^2 g y\right]^{3/2}$$

$$W = W_y + \alpha_x x \qquad = \frac{-1}{3m^2 g}\left[2m\alpha - \alpha_x^2 - 2m^2 g y\right]^{3/2}$$

where $\alpha_x = P_x$

$$S = W_y + \alpha_x x - \alpha t$$

$$\beta = \frac{\partial S}{\partial \alpha} = \frac{-1}{\cancel{3}m^2 g}\frac{\cancel{3}}{\cancel{2}}\left[2m\alpha - \alpha_x^2 - 2m^2 g y\right]^{\frac{1}{2}} \cdot \cancel{2m} - t$$

$$\beta + t = \frac{-1}{mg}\left[2m\alpha - \alpha_x^2 - 2m^2 g y\right]^{\frac{1}{2}}$$

$$\left[mg(\beta + t)\right]^2 = 2m\alpha - \alpha_x^2 - 2m^2 g y$$

$$y = \frac{-m^2 g^2(\beta+t)^2 - \alpha_x^2 + 2m\alpha}{2m^2 g}$$

$$y = -\frac{1}{2}g(\beta+t)^2 - \frac{\alpha_x}{2m^2 g} + \frac{\alpha}{mg} \quad \text{— (1)}$$

$$p_y = \frac{\partial W_y}{\partial y} = -mg(\beta + t)$$

at $t = 0 \Rightarrow P_{y_0} = -mg\beta \Rightarrow \beta = -\frac{P_{y_0}}{mg}$

$$t=0$$

$$\text{(1)} \Rightarrow y(0) = -\tfrac{1}{2} g \beta^2 - \frac{\alpha_x}{2m^2 g} + \frac{\alpha}{mg}$$

$$y_0 = -\tfrac{1}{2} g \cdot \frac{P_{y_0}^2}{m^2 g^2} - \frac{\alpha_x}{2m^2 g} + \frac{\alpha}{mg}$$

$$\text{(1)} \Rightarrow y(t) = \underbrace{-\tfrac{1}{2} g \beta^2}_{-\frac{1}{2} g \frac{P_{y_0}^2}{m^2 g^2}} - \tfrac{1}{2} g t^2 - g \beta t - \frac{\alpha_x}{2m^2 g} + \frac{\alpha}{mg}$$

$$= y_0$$

$$y(t) = y_0 + \frac{P_{y_0}}{m} t - \tfrac{1}{2} g t^2 \quad ✓$$

$$\beta' = \frac{\partial S}{\partial \alpha_x} = -\frac{1}{\cancel{3} m^2 g \cancel{2}} \frac{\cancel{3}}{}\left[2m\alpha - \alpha_x^2 - 2m^2 g y\right]^{\frac{1}{2}} \cdot -\cancel{2}\alpha_x + x$$

$$x = \beta' - \frac{\alpha_x}{m^{\cancel{2}} \cancel{g}} \underbrace{\left[2m\alpha - \alpha_x^2 - 2m^2 g y\right]^{\frac{1}{2}}}_{-\cancel{m}\cancel{g}(\beta + t)}$$

at $t=0$

$$x_0 = \beta' + \frac{\alpha_x}{m} \beta \Rightarrow \beta' = x_0 - \frac{\alpha_x}{m} \beta$$

$$x(t) = x_0 - \frac{\alpha_x \cancel{\beta}}{\cancel{m}} + \frac{\alpha_x}{m}(\cancel{\beta} + t)$$

$$x(t) = x_0 + \frac{\alpha_x}{m} t \quad ✓, \qquad P_{x_0} = \alpha_x = \text{constant}$$

Exercise 9.05

A particle of mass m moves in a three-dimensional isotropic oscillator well

$$V = \tfrac{1}{2}m\omega^2(x^2+y^2+z^2) = \tfrac{1}{2}m\omega^2(\rho^2+z^2) = \tfrac{1}{2}m\omega^2 r^2.$$

(a) Separate the Hamilton-Jacobi equation in cartesian coordinates (x,y,z), find the action variables, and express the Hamiltonian in terms of these. Find the frequencies $(\omega_x,\omega_y,\omega_z)$.
(b) Separate the Hamilton-Jacobi equation in cylindrical coordinates (ρ,ϕ,z), find the action variables, and express the Hamiltonian in terms of these. Find the frequencies $(\omega_\rho,\omega_\phi,\omega_z)$.
(c) Separate the Hamilton-Jacobi equation in spherical polar coordinates (r,θ,ϕ), find the action variables, and express the Hamiltonian in terms of these. Find the frequencies $(\omega_r,\omega_\theta,\omega_\phi)$.

Solution

(a) Writing down the Hamilton-Jacobi equation in cartesian coordinates and solving it by separation of variables proceeds as in Exercise 8.06. The momenta conjugate to x, y, and z are

$$p_x = \sqrt{2m\alpha_x - m^2\omega^2x^2}, \quad p_y = \sqrt{2m\alpha_y - m^2\omega^2y^2}, \quad p_z = \sqrt{2m\alpha_z - m^2\omega^2z^2},$$

where the separation constants α_x, α_y, and α_z are the energies associated with each degree of freedom, with $\alpha_x+\alpha_y+\alpha_z = E$. The action variables are then

$$I_x = \frac{1}{2\pi}\oint\sqrt{2m\alpha_x - m^2\omega^2x^2}\,dx = \alpha_x/\omega,$$
$$I_y = \frac{1}{2\pi}\oint\sqrt{2m\alpha_y - m^2\omega^2y^2}\,dy = \alpha_y/\omega,$$
$$I_z = \frac{1}{2\pi}\oint\sqrt{2m\alpha_z - m^2\omega^2z^2}\,dz = \alpha_z/\omega.$$

The total energy E, in terms of the action variables, is

$$E = \omega(I_x+I_y+I_z),$$

and the frequencies associated with the three degrees of freedom are

$$\omega_x = \frac{\partial E}{\partial I_x} = \omega, \quad \omega_y = \frac{\partial E}{\partial I_y} = \omega, \quad \omega_z = \frac{\partial E}{\partial I_z} = \omega.$$

(b) The Hamilton-Jacobi equation in cylindrical coordinates is

$$\frac{1}{2m}\left(\left(\frac{\partial W}{\partial\rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial W}{\partial\phi}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2\right) + \frac{1}{2}m\omega^2\left(\rho^2+z^2\right) = E.$$

We try a solution of the form

$$W = R(\rho) + \Phi(\phi) + Z(z).$$

The Hamilton-Jacobi equation becomes

$$2m\rho^2\left[\frac{1}{2m}\left(\left(\frac{dR}{d\rho}\right)^2 + \left(\frac{dZ}{dz}\right)^2\right) + \frac{1}{2}m\omega^2\left(\rho^2+z^2\right) - E\right] = -\left(\frac{d\Phi}{d\phi}\right)^2.$$

The left-hand side is a function only of ρ and z, whereas the right-hand side is a function only of ϕ. Both sides must equal a constant. We set

$$2m\rho^2\left[\frac{1}{2m}\left(\left(\frac{dR}{d\rho}\right)^2 + \left(\frac{dZ}{dz}\right)^2\right) + \frac{1}{2}m\omega^2\left(\rho^2+z^2\right) - E\right] = -L_z^2, \qquad \left(\frac{d\Phi}{d\phi}\right)^2 = L_z^2,$$

where the separation constant L_z turns out to be the z-component of the angular momentum. The ρz-equation can be written

$$\left(\frac{1}{2m}\left(\frac{dR}{d\rho}\right)^2 + \frac{1}{2}m\omega^2\rho^2 + \frac{L_z^2}{2m\rho^2}\right) + \left(\frac{1}{2m}\left(\frac{dZ}{dz}\right)^2 + \frac{1}{2}m\omega^2z^2\right) = E.$$

The first term on the left is a function only of ρ, the second term is a function only of z, and their sum is the constant E. Each term must equal a constant. We set

$$\frac{1}{2m}\left(\frac{dR}{d\rho}\right)^2 + \frac{1}{2}m\omega^2\rho^2 + \frac{L_z^2}{2m\rho^2} = E_\perp, \qquad \frac{1}{2m}\left(\frac{dZ}{dz}\right)^2 + \frac{1}{2}m\omega^2z^2 = E_\parallel,$$

where $E_\perp + E_\parallel = E$. The separation constants $E_\perp$ and $E_\parallel$ are the energies associated with the "perpendicular" (or $\rho\phi$) and "parallel" (or z) motions. The variables are now completely separated, and we can read off the momenta conjugate to ρ, ϕ, and z,

$$p_\rho = \frac{\partial W}{\partial\rho} = \frac{dR}{d\rho} = \sqrt{2mE_\perp - m^2\omega^2\rho^2 - L_z^2/\rho^2},$$
$$p_\phi = \frac{\partial W}{\partial\phi} = \frac{d\Phi}{d\phi} = L_z,$$

$$p_z = \frac{\partial W}{\partial z} = \frac{dZ}{dz} = \sqrt{2mE_\parallel - m^2\omega^2 z^2}\,.$$

The action variable associated with the ρ degree of freedom is

$$I_\rho = \frac{1}{\pi}\int_{\rho_{min}}^{\rho_{max}} \sqrt{2mE_\perp - m^2\omega^2\rho^2 - L_z^2/\rho^2}\, d\rho$$
$$= \frac{1}{\pi}\int_{\rho_{min}}^{\rho_{max}} \left[\frac{mE_\perp - m^2\omega^2\rho^2}{\sqrt{2mE_\perp - m^2\omega^2\rho^2 - L_z^2/\rho^2}} + \frac{mE_\perp}{\sqrt{\ }} - \frac{L_z^2}{\rho^2\sqrt{\ }}\right] d\rho$$

where the square root is taken positive and where ρ_{min} and ρ_{max} are the inner and outer turning radii (the values of ρ which make the square root zero). The first term is

$$\frac{1}{\pi}\int_{\rho_{min}}^{\rho_{max}} \frac{mE_\perp - m^2\omega^2\rho^2}{\sqrt{\ }}\, d\rho = \frac{1}{2\pi}\int_{\rho_{min}}^{\rho_{max}} d\left(\rho\sqrt{\ }\right) = 0\,,$$

since the square root is zero at the turning radii. The second term is

$$\frac{1}{\pi}\int_{\rho_{min}}^{\rho_{max}} \frac{mE_\perp}{\sqrt{\ }}\, d\rho = \frac{E_\perp}{2\pi\omega}\int_{\rho_{min}}^{\rho_{max}} \frac{2\rho\, d\rho}{\sqrt{\frac{2E_\perp}{m\omega^2}\rho^2 - \rho^4 - \frac{L_z^2}{m^2\omega^2}}}$$
$$= \frac{E_\perp}{2\pi\omega}\int_{\rho_{min}}^{\rho_{max}} \frac{2\rho\, d\rho}{\sqrt{\frac{E_\perp^2}{m^2\omega^4}\left(1 - \frac{\omega^2 L^2}{E_\perp^2}\right) - \left(\frac{E_\perp}{m\omega^2} - \rho^2\right)^2}}$$

The integration can be performed by setting

$$\frac{E_\perp}{m\omega^2} - \rho^2 = \frac{E_\perp}{m\omega^2}\sqrt{1 - \frac{\omega^2 L_z^2}{E_\perp^2}}\cos\alpha\,, \qquad 2\rho\, d\rho = \frac{E_\perp}{m\omega^2}\sqrt{1 - \frac{\omega^2 L_z^2}{E_\perp^2}}\sin\alpha\, d\alpha\,.$$

As ρ increases from ρ_{min} to ρ_{max} with the square root positive, α increases from 0 to π and we must take $\sqrt{\sin^2\alpha} = \sin\alpha$. The second term then gives $E_\perp/2\omega$. The third term is

$$-\frac{1}{\pi}\int_{\rho_{min}}^{\rho_{max}} \frac{L_z^2}{\rho^2\sqrt{\ }}\, d\rho = -\frac{|L_z|}{2\pi}\int_{\rho_{min}}^{\rho_{max}} \frac{2d\rho/\rho^3}{\sqrt{\frac{2mE_\perp}{L_z^2}\frac{1}{\rho^2} - \frac{m^2\omega^2}{L_z^2} - \frac{1}{\rho^4}}}$$
$$= -\frac{|L_z|}{2\pi}\int_{\rho_{min}}^{\rho_{max}} \frac{2d\rho/\rho^3}{\sqrt{\frac{m^2E_\perp^2}{L_z^4}\left(1 - \frac{\omega^2 L_z^2}{E_\perp^2}\right) - \left(\frac{1}{\rho^2} - \frac{mE_\perp}{L_z^2}\right)^2}}$$

The integration can be performed by setting

$$\frac{1}{\rho^2} - \frac{mE_\perp}{L_z^2} = \frac{mE_\perp}{L_z^2}\sqrt{1 - \frac{\omega^2 L_z^2}{E_\perp^2}}\cos\beta\,, \qquad \frac{2d\rho}{\rho^3} = \frac{mE_\perp}{L_z^2}\sqrt{1 - \frac{\omega^2 L_z^2}{E_\perp^2}}\sin\beta\, d\beta\,.$$

As ρ increases from ρ_{min} to ρ_{max} with the square root positive, β increases from 0 to π and we must take $\sqrt{\sin^2\beta} = \sin\beta$. The third term then gives $-|L_z|/2$. Putting these results together, we find the action variable associated with the ρ degree of freedom

$$I_\rho = \frac{E_\perp}{2\omega} - \frac{|L_z|}{2}\,.$$

The action variable associated with the ϕ degree of freedom is

$$I_\phi = \frac{1}{2\pi}\oint L_z\, d\phi = |L_z|\,.$$

The action variable associated with the z degree of freedom is

$$I_z = \frac{1}{\pi}\int_{z_{min}}^{z_{max}} \sqrt{2mE_\parallel - m^2\omega^2 z^2}\, dz = E_\parallel/\omega\,.$$

These results can be inverted to express the energy in terms of the action variables,

$$E = \omega(2I_\rho + I_\phi + I_z)\,.$$

The frequencies associated with the three degrees of freedom are then

$$\omega_\rho = \frac{\partial E}{\partial I_\rho} = 2\omega\,, \qquad \omega_\phi = \frac{\partial E}{\partial I_\phi} = \omega\,, \qquad \omega_z = \frac{\partial E}{\partial I_z} = \omega\,.$$

4.

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) + \frac{q}{2}Br^2\dot{\theta}.$$

a) If we go to Cartesian coordinates we find

$$m\frac{d\vec{v}}{dt} = q\vec{v} \times \vec{B},$$

which shows that $\vec{v}$ has constant magnitude and just changes direction. It is clear that $v_z = const$ since $\vec{B} = (0, 0, B)$, and that the particle moves in a circle if we project the motion onto the xy-plane.

b) The EOM for r is

$$m\ddot{r} - r\dot{\theta}(m\dot{\theta} + qB) = 0.$$

In a coordinate system where $r = 0$ is the center of the orbit, we have $\dot{r} = 0 = \ddot{r}$. Hence

$$\dot{\theta} = -qB/m$$

c)

$$H = \frac{1}{2m}[p_r^2 + p_z^2 + (p_\theta - qBr^2/2)^2/r^2]$$

where $p_\theta = mr^2\dot{\theta} + qBr^2/2$. Both p_θ and p_z are conserved.

d) The energy $E = H$ is conserved and θ and z are cyclic. Thus $S = W - Et$, with

$$W = W' + p_\theta\theta + p_z z.$$

Ansatz: $W' = W_r(r)$

We find

$$E = \frac{1}{2m}\left[\frac{\partial W_r}{\partial r}^2 + p_z^2 + (p_\theta - qBr^2/2)^2/r^2\right]$$

and thus W_r indeed depends only on r and the constants E, p_z and p_θ:

$$W_r = \int \sqrt{2mE - [p_z^2 + (p_\theta - qBr^2/2)^2/r^2]}dr$$

e) Since the problem is completely separable we can define 3 action variables, that are all adiabatic invariants. Here we need only J_θ.

$$J_\theta = \oint p_\theta d\theta = 2\pi p_\theta$$

since $p_\theta = const$. If we combine $\dot{\theta} = -qB/m$ and $p_\theta = mr^2\dot{\theta} + qBr^2/2$ from b) and c) we find

$$p_\theta = -qBr^2/2.$$

If B is slowly changed, J_θ and thus p_θ is constant. I.e. $-qBr_0^2/2 = -q4Br^2/2$. So the radius of the helix is $r = r_0/2$ after B is increased by a factor of 4.

Similarly $p_z = const$ and $J_z = 2\pi p_z$. Hence $\dot{z}$ does not change.

From $\dot{\theta} = -qB/m$, we know that $\dot{\theta} = 4\dot{\theta}_0$.