Mechanics - PHY 6247

Solutions to HW 6

——————————— P1 ———————————————

part a)

$H = \frac{p^2}{2m} + A(q)p + B(q)$

the generalized velocity is : $\frac{\partial H}{\partial p_r} = \dot{q}_r$

(1) $\dot{q} = \frac{\partial H}{\partial p} = \frac{\partial}{\partial p} = \frac{p}{m} + A(q)$

part b)

$L = \sum p_r \dot{q}_r - H$

$L = p\dot{q} - \left[\frac{p^2}{2m} + A(q)p + B(q)\right]$

from (1) $p = m\,(\dot{q} - A)$

$L = -\frac{p^2}{2m} + p(\dot{q} - A) - B$

$L = -\frac{p^2}{2m} + \frac{p^2}{m} - B$

$L = \frac{p^2}{2m} - B$

now replace $p = (\dot{q} - A)m$ from (1)

$L = \frac{(\dot{q}-A)^2 m^2}{2m} - B$

$L = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}mA^2(q) - mA(q)\dot{q} - B(q)$

——————————— P2———————————————————

–P2.a–

$L' = L + \frac{d\Lambda}{dt}$

(1)$\frac{d\Lambda}{dt} = \sum_i \frac{\partial \Lambda}{\partial q_i}\dot{q}_i + \frac{\partial \Lambda}{\partial t}$

$p = \frac{\partial L}{\partial \dot{q}}$

$p' = \frac{\partial L'}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}}\left(L + \sum_i \frac{\partial \Lambda}{\partial q_i}\dot{q}_i + \frac{\partial \Lambda}{\partial t}\right)$

$p' = \frac{\partial L}{\partial \dot{q}} + \frac{\partial}{\partial \dot{q}}\sum_i \frac{\partial \Lambda}{\partial q_i}\dot{q}_i + \frac{\partial}{\partial \dot{q}}\left(\frac{\partial \Lambda}{\partial t}\right)$

Now by the definition $\Lambda = \Lambda(q,t) \Rightarrow \frac{\partial}{\partial \dot{q}}\left(\frac{\partial \Lambda}{\partial t}\right) = 0$ (2)

(3) $p'_i = p_i + \frac{\partial \Lambda}{\partial q_i}$

which is the requested relationship.

–P2.b–

prob 2.b)

$H = \sum p_r \dot{q}_r - L = p\dot{q} - L$

$H' = \sum p'_r \dot{q}_r - L' = p'\dot{q} - L'$

replace (4) $p' = p + \frac{\partial \Lambda}{\partial q}$

$H' = \left(p + \frac{\partial \Lambda}{\partial q}\right)\dot{q} - L - \frac{d\Lambda}{dt}$

$H' = (p\dot{q} - L) + \frac{\partial \Lambda}{\partial q}\dot{q} - \left(\frac{\partial \Lambda}{\partial q}\dot{q} + \frac{\partial \Lambda}{\partial t}\right)$

(5) $H' = H - \frac{\partial \Lambda}{\partial t}$

prob 2.c

Hamiltonian eq of motions are

$\frac{\partial H'}{\partial q_i} = -\dot{p}'_i = \frac{dp'_i}{dt}$

subst $p' = p + \frac{\partial \Lambda}{\partial q}$

$\frac{\partial H'(q,p')}{\partial q_i} = \frac{\partial}{\partial q_i}(H(q,p(p',q)) - \frac{\partial \Lambda}{\partial t}) = \frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial p_j}\frac{\partial p_j(p',q)}{\partial q_i} - \frac{\partial}{\partial q_i}\left(\frac{\partial \Lambda}{\partial t}\right) = \frac{\partial H}{\partial q_i} + \dot{q}_j(-\frac{\partial}{\partial q_i}\frac{\partial \Lambda}{\partial q_j}) - \frac{\partial}{\partial q_i}\left(\frac{\partial \Lambda}{\partial t}\right)$

$-\dot{p}'_i = -\frac{d}{dt}\left(p_i + \frac{\partial \Lambda}{\partial q_i}\right) = -\dot{p}_i - \frac{\partial}{\partial q_i}\frac{\partial \Lambda}{\partial q_j}\dot{q}_j - \frac{\partial}{\partial q_i}\left(\frac{\partial \Lambda}{\partial t}\right)$

Thus $\frac{\partial H'(q,p')}{\partial q_i} = -\dot{p}'_i$ gives $\frac{\partial H}{\partial q_i} = -\dot{p}_i$, which means the EOM for $\dot{p}$ is unchanged!

$\frac{\partial H'(q,p')}{\partial p'_i} = \frac{\partial}{\partial p'_i}(H(q,p(p',q)) - \frac{\partial \Lambda}{\partial t}) = \frac{\partial H}{\partial p_j}\frac{\partial p_j(p',q)}{\partial p'_i} - \frac{\partial}{\partial p'_i}\left(\frac{\partial \Lambda}{\partial t}\right) = \frac{\partial H}{\partial p_j}(\delta_{ij} - \frac{\partial}{\partial p'_i}\frac{\partial \Lambda}{\partial q_j}) - \frac{\partial}{\partial p'_i}\left(\frac{\partial \Lambda}{\partial t}\right).$

Now note that $\frac{\partial}{\partial p'_i}\Lambda = 0$. Thus$\frac{\partial H'(q,p')}{\partial p'_i} = \frac{\partial H}{\partial p_i}$, so that $\frac{\partial H}{\partial p'_i} = \dot{q}_i$ gives $\frac{\partial H}{\partial p_i} = \dot{q}_i$, which means the EOM for $\dot{q}$ is unchanged!

——————————— P3———————————

$L = \frac{m}{2}\left[(\dot{x}^2 + \dot{y}^2 + \dot{x}^2) + 2\omega(x\dot{y} - y\dot{x}) + \omega^2(x^2 + y^2\right]$

In the eqn above all symbols are meant to be primed as for instance $\dot{x}^2 = \dot{x}'^2$.

$H = \sum \dot{q}_i p_i - L$ given $p_i = \frac{\partial L}{\partial \dot{q}_i}$

$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - m\omega y$

$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + m\omega x$

$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$

$H = \dot{x}\,(m\dot{x} - m\omega y) + \dot{y}\,(m\dot{y} + m\omega x) + m\dot{z}^2 - \left\{\frac{m}{2}\left[(\dot{x}^2 + \dot{y}^2 + \dot{x}^2) + 2\omega(x\dot{y} - y\dot{x}) + \omega^2(x^2 + y^2\right]\right\}$

$H = m(\dot{x}^2 + \dot{y}^2 + z^2)\left[1 - \frac{1}{2}\right] + m\omega(x\dot{y} - yx)(1 - 1) - \frac{1}{2}m\omega^2(x^2 + y^2)$

$H = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + z^2) - \frac{1}{2}m\omega^2(x^2 + y^2)$

substitute the following into the above expression for H

$\frac{p_x}{m} + \omega y = \dot{x}$

$\frac{p_y}{m} - \omega x = \dot{y}$

$\frac{p_z}{m} = \dot{z}$

$H = \frac{1}{2}m\left[\left(\frac{p_x}{m} + \omega y\right)^2 + \left(\frac{p_y}{m} - \omega x\right)^2 + \frac{p_z^2}{m}\right] - \frac{1}{2}m\omega^2(x^2 + y^2)$

$H = \frac{1}{2}\left(\frac{p_x^2}{m} + \frac{p_z^2}{m} + \frac{p_z^2}{m}\right) + \frac{1}{2}m\omega^2\left(x^2 + y^2\right) - \frac{1}{2}m\omega^2(x^2 + y^2) + \omega\left(p_x y - p_y x\right)$

$H = \frac{1}{2}\left(\frac{p_x^2}{m} + \frac{p_z^2}{m} + \frac{p_z^2}{m}\right) + \omega\left(p_x y - p_y x\right)$

——————————— P4 ———————————————————

The Lagrangian is

$L = \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 \left[l_1\dot{\theta}_1^2 + l_2\dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1\dot{\theta}_2 cos(\theta_2 - \theta_1)\right] + m_1 g l_1 cos\theta_1 + m_2 g \left(l_1 cos\theta_1 + l_2 cos\theta_2\right)$

$\frac{\partial L}{\partial \dot{\theta_1}} = p_{\theta_1} = (m_1 + m_2)\, l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 cos(\theta_2 - \theta_1)$

$\frac{\partial L}{\partial \dot{\theta_2}} = p_{\theta_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 cos(\theta_2 - \theta_1)$

$l_2\dot{\theta}_2 = \frac{p_{\theta_2}}{m_2 l_2} - l_1\dot{\theta}_1 cos(\theta_2 - \theta_1)$

$p_{\theta_1} = (m_1 + m_2)\, l_1^2\dot{\theta}_1 + m_2 l_1 \left[\frac{p_{\theta_2}}{m_2 l_2} - l_1\dot{\theta}_1 cos^2(\theta_2 - \theta_1)\right]$

$\dot{\theta}_1 = \frac{l_2 p_{\theta_1} - l_1 p_{\theta_2} cos(\theta_2 - \theta_1)}{l_1^2 l_2 \{m_1 + m_2[1 - cos^2(\theta_2 - \theta_1)]\}}$

$l_2\dot{\theta}_2 = \frac{p_{\theta_2}}{m_2 l_2} - l_1 \left[\frac{l_2 p_{\theta_1} - l_1 p_{\theta_2} cos(\theta_2 - \theta_1)}{l_1^2 l_2 \{m_1 + m_2 sin(\theta_2 - \theta_1)\}}\right] cos(\theta_2 - \theta_1)$

$\dot{\theta}_2 = \frac{p_{\theta_2}}{m_2 l_2^2} - \left[\frac{l_2 l_1 p_{\theta_1} cos(\theta_2 - \theta_1) - l_1^2 p_{\theta_2} cos^2(\theta_2 - \theta_1)}{l_1^2 l_2^2 \{m_1 + m_2 sin^2(\theta_2 - \theta_1)\}}\right]$

$\dot{\theta}_2 = \frac{l_1^2 p_{\theta_2}\left\{m_1 + m_2\left[sin^2(\theta_2 - \theta_1)\right]\right\} -}{m_2 l_1^2 l_2^2 \{m_1 + m_2[1 - cos^2(\theta_2 - \theta_1)]\}} - \frac{m_2 l_2 l_1 p_{\theta_1} cos(\theta_2 - \theta_1) - m_2 l_1^2 p_{\theta_2} cos^2(\theta_2 - \theta_1)}{m_2 l_1^2 l_2^2 \{m_1 + m_2[1 - cos^2(\theta_2 - \theta_1)]\}}$

$\dot{\theta}_2 = \frac{l_1^2 p_{\theta_2} m_1 + m_2 l_1^2 p_{\theta_2} sin^2(\theta_2 - \theta_1)}{m_2 l_1^2 l_2^2 \{m_1 + m_2[1 - cos^2(\theta_2 - \theta_1)]\}} - \frac{m_2 l_2 l_1 p_{\theta_1} cos(\theta_2 - \theta_1) - m_2 l_1^2 p_{\theta_2} cos^2(\theta_2 - \theta_1)}{m_2 l_1^2 l_2^2 \{m_1 + m_2[1 - cos^2(\theta_2 - \theta_1)]\}}$

$\dot{\theta}_2 = \frac{l_1^2 p_{\theta_2} m_1 + m_2 l_1^2 p_{\theta_2} sin^2(\theta_2 - \theta_1) + m_2 l_1^2 p_{\theta_2} cos^2(\theta_2 - \theta_1)}{m_2 l_1^2 l_2^2 \{m_1 + m_2[1 - cos^2(\theta_2 - \theta_1)]\}} - \frac{m_2 l_2 l_1 p_{\theta_1} cos(\theta_2 - \theta_1)}{m_2 l_1^2 l_2^2 \{m_1 + m_2[1 - cos^2(\theta_2 - \theta_1)]\}}$

$\dot{\theta}_2 = \frac{l_1 p_{\theta_2} m_1 + m_2 l_1 p_{\theta_2} - m_2 l_2 p_{\theta_1} cos(\theta_2 - \theta_1)}{m_2 l_1 l_2^2 \{m_1 + m_2[1 - cos^2(\theta_2 - \theta_1)]\}}$

$\dot{\theta}_2 = \frac{(m_1 + m_2) l_1 p_{\theta_2} - m_2 l_2 p_{\theta_1} cos(\theta_2 - \theta_1)}{m_2 l_1 l_2^2 (m_1 + m_2 sin^2(\theta_2 - \theta_1))}$

$\dot{\theta}_1 = \frac{l_2 p_{\theta_1} - l_1 p_{\theta_2} cos(\theta_2 - \theta_1)}{l_1^2 l_2 (m_1 + m_2 sin^2(\theta_2 - \theta_1))}$

$H = \dot{\theta}_1 p_{\theta_1} + \dot{\theta}_2 p_{\theta_2} - L$

for convenience, from this point on $p_{\theta_1} = p_1$; $p_{\theta_2} = p_2$

$H = -g l_1 \left(m1 + m2\right) cos\theta_1 - g m_2 l_2 cos\theta_2 + \frac{(m_1 + m_2) l_1^2 p_2^2 + m_2 p_1^2 l_2^2 - 2 m_2 p_1 p_2 cos(\theta_1 - \theta_2)}{2 l_1^2 l_2^2 m_2 \{m_1 + m_2 sin^2(\theta_1 - \theta_2)\}}$

$-\frac{\partial H}{\partial p_1} = \frac{\left[\dot{l}_1 p_2 cos(\theta_1 - \theta_2) - l_2 p_1\right]}{l_1^2 l_2 [m_1 + m_2 sin^2(\theta_1 - \theta_2)]}$

$-\frac{\partial H}{\partial p_2} = \frac{(m1 + m2) l_1 p_2 - m_2 l_2 p_1 cos(\theta_1 - \theta_2)}{l_1 l_2^2 m_2 [m_1 + m_2 sin^2(\theta_1 - \theta_2)]}$

$\frac{\partial H}{\partial \theta_1} = -g l_1 \left(m1 + m2\right) sin\theta_1 - \frac{p_1 p_2 sin(\theta_1 - \theta_2)}{l_1 l_2 [m_1 + m_2 sin^2(\theta_1 - \theta_2)]} + \frac{sin(\theta_1 - \theta_2)\left[(m_1 + m_2) l_1^2 p_2^2 + m_2 p_1^2 l_2^2 - 2 m_2 p_1 p_2 cos(\theta_1 - \theta_2)\right]}{2 l_1^2 l_2^2 [m_1 + m_2 sin^2(\theta_1 - \theta_2)]}$

$\frac{\partial H}{\partial \theta_2} = +g l_2 m_2 sin\theta - \frac{p_1 p_2 sin(\theta_1 - \theta_2)}{l_1 l_2 [m_1 + m_2 sin^2(\theta_1 - \theta_2)]} + \frac{sin(\theta_1 - \theta_2)\left[(m_1 + m_2) l_1^2 p_2^2 + m_2 p_1^2 l_2^2 - 2 m_2 p_1 p_2 cos(\theta_1 - \theta_2)\right]}{2 l_1^2 l_2^2 [m_1 + m_2 sin^2(\theta_1 - \theta_2)]}$ ———

——————— P5———————————————————

$T = \frac{1}{2}m\dot{x}^2$
$U = \frac{1}{2}k(x-l)^2$
$L = T - U = \frac{1}{2}\left[m\dot{x}^2 - k(x-l)^2\right]$
$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$
$\dot{x} = \frac{p_x}{m}$
$H = \sum \dot{q}_i p_i - L = m\dot{x}^2 - \frac{1}{2}\left[m\dot{x}^2 - k(x-l)^2\right]$
$H = \frac{1}{2}\left[m\dot{x}^2 + k(x-l)^2\right]$
$H = \frac{1}{2}\left[\frac{p_x^2}{m} + k(x-l)^2\right]$
Hamilton eq of motion
$-\frac{\partial H}{\partial q_i} = \dot{p}_i\,;\,\frac{\partial H}{\partial p_i} = q_i$
$-\frac{\partial H}{\partial x} = \dot{p}_x = -k(x-l)$
$\frac{\partial H}{\partial p_x} = \dot{x} = \frac{p_x}{m}$

——————————— P6———————————

$L = a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{x}y + \dot{f}y^2\dot{x}\dot{z} + g\dot{y}^2 - k\sqrt{x^2+y^2}$

P6.a) Find the Hamiltonian

(1)$\frac{\partial L}{\partial \dot{x}} = p_x = c\dot{y} + 2a\dot{x} + fy^2\dot{z}$

(2)$\frac{\partial L}{\partial \dot{y}} = p_y = \frac{b}{x} + c\dot{x} + 2g\dot{y}$

(3)$\frac{\partial L}{\partial \dot{z}} = p_z = fy^2\dot{x}$

$H = \sum{}_i \dot{q}_i p_i - L$

$H = \dot{x}(c\dot{y} + 2a\dot{x} + fy^2\dot{z}) + \dot{y}\left(\frac{b}{x} + c\dot{x} + 2g\dot{y}\right) + \dot{z}\left(fy^2\dot{x}\right)$

$-\left[a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{x}y + \dot{f}y^2\dot{x}\dot{z} + g\dot{y}^2 - k\sqrt{x^2+y^2}\right]$

$H = 2a\dot{x}^2 + c\dot{x}\dot{y} + fy^2\dot{x}\dot{z} + b\frac{\dot{y}}{x} + c\dot{x}\dot{y} + 2g\dot{\dot{y}}^2 + fy^2\dot{x}\dot{z}$

$-\left[a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{x}y + \dot{f}y^2\dot{x}\dot{z} + g\dot{y}^2 - k\sqrt{x^2+y^2}\right]$

$H = xP_x + yp_y + p_z\dot{z} - \left[a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{x}\dot{y} + p_z\dot{z} + g\dot{y}^2 - k\sqrt{x^2+y^2}\right]$

$H = a\dot{x}^2 + c\dot{x}\dot{y} + g\dot{y}^2 + fy^2\dot{x}\dot{z} + k\sqrt{x^2+y^2}$

Collect $\dot{x}$

$H = (a\dot{x} + c\dot{y} + fy^2\dot{z})\dot{x} + g\dot{y}^2 + k\sqrt{x^2+y^2}$

$H = (a\dot{x} + c\dot{y} + fy^2\dot{z})\dot{x} + g\dot{y}^2 + k\sqrt{x^2+y^2}$

from (1) $p_x - 2a\dot{x} - fy^2\dot{z} = c\dot{y}$

$H = (a\dot{x} + p_x - 2a\dot{x} - fy^2\dot{z} + fy^2\dot{z})\dot{x} + g\dot{y}^2 + k\sqrt{x^2+y^2}$

$H = (-a\dot{x} + p_x)\dot{x} + g\dot{y}^2 + k\sqrt{x^2+y^2}$

from (2)$\frac{\left(p_y - \frac{b}{x} - c\dot{x}\right)^2}{4g^2} = \dot{y}^2$

$H = \left[-a\left(p_z/fy^2\right) + p_x\right]\left(p_z/fy^2\right) + g\left(p_y - \frac{b}{x} - c\dot{x}\right)^2 + k\sqrt{x^2+y^2}$

from (3) $p_z/fy^2 = \dot{x}$

$H = \left[-a\left(p_z/fy^2\right) + p_x\right]\left(p_z/fy^2\right) + \frac{g}{4g^2}\left[p_y - \frac{b}{x} - c\left(p_z/fy^2\right)\right]^2 + k\sqrt{x^2+y^2}$

$H = -ap_z^2/f^2y^4 + p_xp_z/fy^2 + \frac{1}{4g}\left[p_y - \frac{b}{x} - cp_z/fy^2\right]^2 + k\sqrt{x^2+y^2}$

$\frac{\partial H}{\partial x} = \dot{p}_x = \frac{kx}{\sqrt{x^2+y^2}} + \frac{b}{2gx^2}\left(bp - \frac{b}{x} - \frac{cp_z}{fy^2}\right)$

$\frac{\partial H}{\partial y} = \dot{p}_y = \frac{4ap_z^2}{f^2y^5} + \frac{cp_xp_z^2(p_y - \frac{b}{x} - \frac{cp_z}{fy^2})}{f^2gy^5} - \frac{p_xp_z(p_y - \frac{b}{x} - \frac{cp_z}{fy^2})^2}{2fgy^3} + \frac{ky}{\sqrt{x^2+y^2}}$

b)

$\frac{\partial H}{\partial z} = \dot{p}_z = 0$ hence p_z is conserved. $\frac{\partial H}{\partial t} = 0$ hence H is conserved

P7

$L = \dot{q}_1^2 + \frac{\dot{q}_2^2}{a+bq_1^2} + k_1 q_1^2 + k_2 \dot{q}_1 \dot{q}_2$

$\frac{\partial L}{\partial \dot{q_1}} = p_1 = 2\dot{q}_1 + k_2 \dot{q}_2$

$\frac{\partial L}{\partial \dot{q_2}} = p_2 = \frac{2\dot{q}_2}{a+bq_1^2} + k_2 \dot{q}_1$

solving for $\dot{q}_1, \dot{q}_2$we get

$\dot{q}_1 = \frac{2p_1 - k_2 p_2 (a+bq_1^2)}{4-k_2^2(a+bq_1^2)}$

$\dot{q}_2 = \frac{(k_2 p_1 - 2p_2)\left(a+bq_1^2\right)}{4-k_2^2(a+bq_1^2)}$

$H = \sum_i \dot{q}_i p_i - L$

$H = \frac{p_1^2 + p_2^2\left(a+b^2 q_1\right) + k_2 p_1 p_2 \left(a+b^2 q_1\right)}{4-k_2^2(a+b^2 q_1)} - k_1 q_1^2$

$\frac{\partial H}{\partial q_1} = -\dot{p}_1 = 2k_1 q_1 - \frac{2bq1(k2p_1 - 2p_2)^2}{\left[4-k^2 2\left(a+bq_1^2\right)\right]^2}$

$\frac{\partial H}{\partial q_2} = -\dot{p}_2 = 0$ p2 is conserved

$\frac{\partial H}{\partial p_1} = \dot{q}_1 = \frac{2p_1 - k_2 p_2 \left(a+bq_1^2\right)}{4-k_2^2\left(a+bq_1^2\right)}$

$\frac{\partial H}{\partial p_2} = \dot{q}_2 = \frac{(2p_2 - k_2 p_1)\left(a+bq_1^2\right)}{4-k_2^2\left(a+bq_1^2\right)}$

P8

$(1)\frac{\partial L}{\partial \dot\theta} = p_\theta = I_x\dot\theta$

$(2)\frac{\partial L}{\partial \dot\phi} = p_\phi = (I_x sin^2\theta + I_z cos^2\theta)\,\dot\phi + \dot\psi I_z cos\theta$

$(2)\frac{\partial L}{\partial \dot\psi} = p_\psi = I_z\left(\dot\phi cos\theta + \dot\psi\right)$

$H = \sum_i \dot q_i p_i - L$

$H = \frac{p_\theta^2}{2I_x} + p_\psi^2\left(\frac{1}{2I_z} + \frac{cot^2\theta}{2I_x}\right) + \frac{p_\phi^2}{2I_x}csc^2\theta - \frac{p_\phi p_\psi cot\theta csc\theta}{I_x} + gMrcos\theta$

$\frac{\partial H}{\partial \theta} = -\dot p_\theta = \frac{\left(p_\psi - p_\phi cos\theta\right)\left(p_\phi - p_\psi cos\theta\right)}{I_x sin^3\theta} - Mgrsin\theta$

$\frac{\partial H}{\partial \phi} = -\dot p_\phi = 0$ conserved

$\frac{\partial H}{\partial \psi} = -\dot p_\psi = 0$ conserved

$\frac{\partial H}{\partial p_\theta} = \dot\theta = \frac{p_\theta}{I_x}$

$\frac{\partial H}{\partial p_\phi} = \dot\phi = \frac{\left(p_\phi - p_\psi cos\theta\right)}{I_x sin^2\theta}$ precession rate

$\frac{\partial H}{\partial p_\psi} = \dot\psi = -\frac{p_\psi}{Iz} + \frac{\left(p_\phi - p_\psi cos\theta\right)}{I_x sin^2\theta}cos\theta$ spinning rate