Mechanics - PHY 6247

Solutions to HW 5

————————————HW5 prob 1————————————-

Le M be a symmetric square matrix
be the definiton we can write
(1) $M = M^T$
part (a)
Let the eigenvalue equation be
(2) $M|v_j >= \lambda_j|v_j >$ (here $|v_j >$ and $< v_j|$ is different notation for $\vec{v_j}$ and $\vec{v_j}^\dagger$)
(3) $< v_i|M^T =< v_i|\lambda_i^*$
Let's use $< v_i|$ (1) $|v_j >$ with (2) and (3)
$< v_i|(M^T - M)|v_j >= (\lambda_i^* - \lambda_j) < v_i|v_j >$
(4) $(\lambda_i^* - \lambda_j) < v_i|v_j >= 0$ from (1)
if $i = j$
$(\lambda_i^* - \lambda_i) = 0 \Rightarrow \lambda_i^{*=}\lambda_i$which proves that $\lambda \subseteq R$
part(b)
If $\lambda_j \neq \lambda_i$ Eq. (4) tells us $< v_i|v_j >= 0$. Thus eigenvectors with different eigenvalues are orthogonal. The eigenvectors with identical eigenvalues lie in the subspace that's orthogonal to all the other eigenvectors. Thus they span this subspace. Since any vector in this subspace is also an eigenvector with the same eigenvalue, we simply choose othogonal vecotors in this subspace. Thus in the end all eigenvectors are orthogonal. We will also normalized to length 1.
After this is done we put the eigenvectors $\vec{v_i}$ inside into the columns of matrix

$$D = (\vec{v_1}\vec{v_2}...\vec{v_n})$$

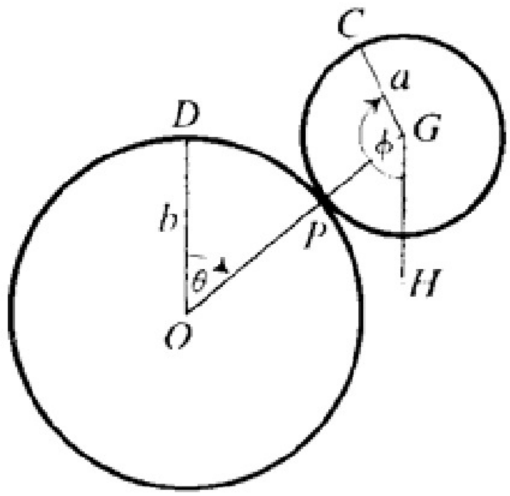
Then

$$MD = (\lambda_1\vec{v_1}\lambda_2\vec{v_2}...\lambda_n\vec{v_n})$$

and thus

$$D^TMD = (\vec{v_1}\vec{v_2}...\vec{v_n})^T(\lambda_1\vec{v_1}\lambda_2\vec{v_2}...\lambda_n\vec{v_n}) = \begin{bmatrix} \lambda_1 & 0 & . \\ 0 & \lambda_2 & . \\ . & . & \lambda_n \end{bmatrix}$$

So D^TMD is indeed diagonal.



The cylinder of radius b rolls without slipping.
Assuming the contact point between the two cylinders as centre of instantaneous rotation, the velocity of the c.m of the cylinder of radius a is $v_{cm} = a\dot{\phi}$
we could as well say that the c.m. rotates about the centre of the larger cylinder as $v_{cm} = (b+a)\dot{\theta}$
consequently $(b+a)\dot{\theta} = a\dot{\phi}$
and $\dot{\phi} = \frac{a+b}{a}\dot{\theta}$
part(b)
$T = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}I\omega^2$
$\dot{r} = 0$
$\omega = \dot{\phi}$
$T = \frac{1}{2}Mr^2\dot{\theta}^2 + \frac{1}{2}I\omega^2$
$T = \frac{1}{2}M(a+b)^2\dot{\theta}^2 + \frac{1}{2}(\frac{1}{2}Ma^2)\dot{\phi}^2 = \frac{3}{4}M(a+b)^2\dot{\theta}^2$
$V = Mg(a+b)cos\theta$
part(c)
$L = T - V = \frac{3}{4}M(a+b)^2\dot{\theta}^2 - Mg(a+b)cos\theta$
$\partial L/\partial\dot{\theta} = \frac{3}{2}M(a+b)^2\dot{\theta}$
$\frac{d}{dt}\left(\frac{3}{2}M(a+b)^2\dot{\theta}\right) = \frac{3}{2}M(a+b)^2\ddot{\theta}$
$\partial L/\partial\theta = Mg(a+b)sin\theta$
$\frac{d}{dt}\left[\partial L/\partial\dot{\theta}\right] - \partial L/\partial\theta = 0 = \frac{3}{2}M(a+b)^2\ddot{\theta} - Mg(a+b)sin\theta$
$\ddot{\theta} = \frac{2}{3}\frac{g}{a+b}sin\theta$

ALTERNATE APPROACH:
One could also incorporate the constraints with the help of Lagrange multipliers, and consider 3 coordinates (r, θ, ϕ). Then

$$L = \frac{M}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{I}{2}\dot{\phi}^2 - Mgr\cos\theta$$

with the constraints $f_1 = r - R$ and $f_2 = \phi - \frac{R}{a}\theta$, where we have defined $R = a + b$. Then

$$I = \int (L(q, \dot{q}) + \lambda_A f_A(q))dt$$

and its variation gives:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} + \lambda_A \frac{\partial f_A}{\partial q_i}$$

Here we get

$$M\ddot{r} = Mr\dot{\theta}^2 - Mg\cos\theta + \lambda_1 \qquad (1a)$$

$$Mr^2\ddot{\theta} + 2Mr\dot{r}\dot{\theta} = Mgr\sin\theta - \frac{R}{a}\lambda_2 \qquad (2a)$$

$$I\ddot{\phi} = \lambda_2 \qquad (3a)$$

Insert (3a) into (2a) and use $\ddot{\phi} = \frac{R}{a}\ddot{\theta}$ (which comes from $f_2 = 0$) to get

$$Mr^2\ddot{\theta} + 2Mr\dot{r}\dot{\theta} = Mgr\sin\theta - \frac{R^2}{a^2}I\ddot{\theta}. \qquad (2b)$$

From $f_1 = 0$ we get $r = R$, $\dot{r} = 0$, and thus

$$MR^2\ddot{\theta} = MgR\sin\theta - \frac{R^2}{a^2}I\ddot{\theta}. \qquad (2c)$$

If we set I to the same value as before we get the same EOM for θ.
But what is nice that we can also figure out WHEN the top cylinder would loose contact with the bottom one. At the last point of contact the constraints are still valid so $r = R$ and $\ddot{r} = 0$. Thus (1a) becomes

$$\lambda_1 = Mg\cos\theta - MR\dot{\theta}^2. \qquad (1b)$$

Now, at the last point of contact the force of constraint λ_1 (here the normal force) goes to zero. So the angle of last contact is determined by

$$Mg\cos\theta = MR\dot{\theta}^2. \qquad (1c)$$

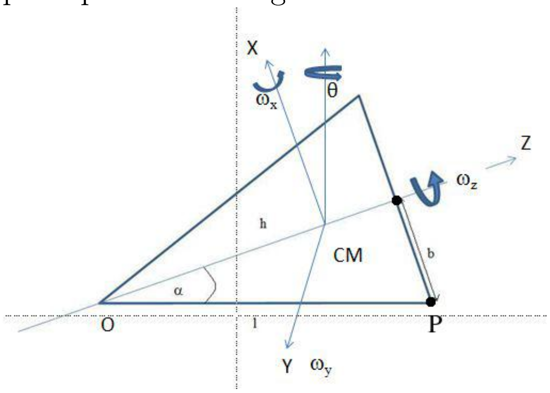
Of course $\dot{\theta}$ depends on initial conditions $(\theta(t=0), \dot{\theta}(t=0))$. But if we insert a particular solution $\theta(t)$, we can find the θ when contact is lost.
We can also use the conserved

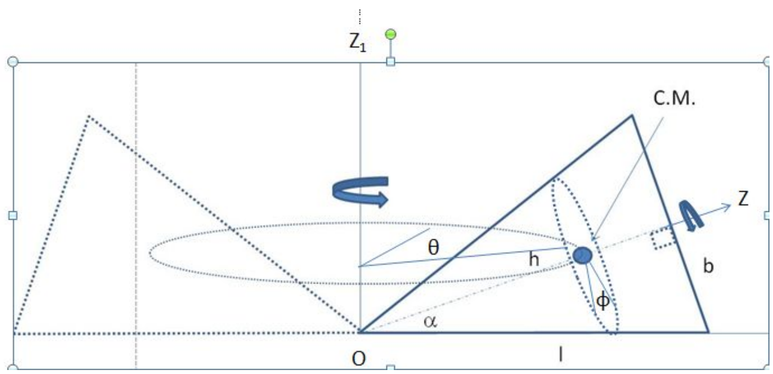
$$E = \frac{1}{2}(MR^2 + I\frac{R^2}{a^2})\dot{\theta}^2 + Mgr\cos\theta,$$

solve for $\dot{\theta}^2$, and insert this $\dot{\theta}^2$ into (1c) to find the angle of last conatct.

——————————HW5 prob 3——————————-

Let $T_{cm} = \frac{1}{2}Mv_{cm}^2$ be the kinetic energy partdue to CM motion. Then $T = T_{cm} + \vec{\omega} \cdot I\vec{\omega}$ where $\vec{\omega}$ is the due to rotation about the CM where the CM is at the origin. Here I is a moment of inertia tensor for rotations about an axis through the center of mass (CM).
Let the principal axes be denoted by X, Y, Z, and assume that the Z axis is parallel to the principal axis through O and CM





The angular velocity $\vec{\omega}$ is due to changes in θ and ϕ (see picture for angles). Thus

$$\vec{\omega} = \dot{\theta}\hat{z} + \dot{\phi}\hat{Z}.$$

The projections onto the body fixed principal axes $\omega_X, \omega_Y, \omega_Z$ are then

$$\omega_Z = \hat{Z} \cdot \vec{\omega} = \dot{\theta}\hat{z} \cdot \hat{Z} + \dot{\phi} = \dot{\theta}\cos(\pi/2 - \alpha) + \dot{\phi} = \dot{\phi} + \dot{\theta}\sin\alpha$$

$$\omega_X = \hat{X} \cdot \vec{\omega} = \dot{\theta}\hat{z} \cdot \hat{X} = \dot{\theta}\cos\alpha\cos\phi$$

$$\omega_Y = \hat{Y} \cdot \vec{\omega} = \dot{\theta}\hat{z} \cdot \hat{Y} = \dot{\theta}\cos\alpha\sin\phi$$

(1) $T = T_{cm} + \frac{1}{2}(I_X\omega_X^2 + I_Y\omega_Y^2 + I_Z\omega_Z^2) = T_{cm} + \frac{1}{2}(I_X(\omega_X^2 + \omega_Y^2) + I_Z\omega_Z^2)$
Here $I_Y = I_X$, because of axisymmetry.
Next note that the cone is rolling without slipping. This means that the point P is instantanously at rest, i.e.

$$l\dot{\theta} + b\dot{\phi} = 0.$$

From the picture we see that $b/l = \sin\alpha$. Thus we get the relationship between $\theta$and ϕ
(2) $\dot{\phi} = -\frac{\dot{\theta}}{sin\alpha}$
(3) $\phi = -\frac{\theta}{sin\alpha}$
[Side note: Insert (2) into expressions for the omega components:

$$\omega_Z = \dot{\phi} + \dot{\theta}\sin\alpha = -\dot{\theta}\cos^2\alpha/\sin\alpha$$

$$\omega_X = \dot{\theta}\cos\alpha\cos\phi$$

$$\omega_Y = \dot{\theta}\cos\alpha\sin\phi$$

Consider $\phi = 0$ and calculate $\vec{\omega} \cdot \hat{x}$ Then

$$\vec{\omega}\cdot\hat{x} = (\dot{\theta}\cos\alpha\hat{X} - \dot{\theta}(\cos^2\alpha/\sin\alpha)\hat{Z})\cdot\hat{x} = \dot{\theta}\cos\alpha(-\sin\alpha) - \dot{\theta}(\cos^2\alpha/\sin\alpha)(\cos\alpha) = -\dot{\theta}(\cos\alpha/\sin\alpha)$$

At the same time $\omega^2 = \omega_X^2 + \omega_Y^2 + \omega_Z^2 = \dot{\theta}^2 \cos^2\alpha / \sin^2\alpha$. This shows that $\vec{\omega}$ is along $\hat{x}$, i.e. along the line OP.
I.e. we could have included the no sliping condition by assuming that the cone rotates about OP at any instant. If we now draw the triangle coming from $\vec{\omega} = \dot{\theta}\hat{z} + \dot{\phi}\hat{Z}$ and note that $\vec{\omega}$ is along OP, we immediately see that $|\dot{\theta}|/|\omega| = \tan\alpha$. I.e. for $\phi = 0$ we could have found the ω components from:

$$\omega_X = \omega \sin\alpha$$

$$\omega_Z = -\omega \cos\alpha$$

$$\omega = \dot{\theta}/\tan\alpha$$

]
Also
$I_Z = \frac{3}{10}Mb^2 = \frac{3}{10}Mh^2\tan^2\alpha$
$I_X = I_Y = \frac{3}{80}M(4b^2 + h^2)$
$T = T_{cm} + \frac{1}{2}\left[I_X(\omega_X^2) + I_Z\omega_Z^2\right]$
$T = T_{cm} + \frac{1}{2}\left[\frac{3}{80}M(4b^2+h^2)\dot{\theta}^2 cos^2\alpha + \frac{3}{10}Mb^2\left(\dot{\phi} + \dot{\theta}sin\alpha\right)^2\right]$
replace (3) $\phi = -\frac{\theta}{sin\alpha}$ into the expression above
$T = T_{cm} + \frac{1}{2}\left[\frac{3}{20}Mb^2\dot{\theta}^2cos^2\alpha + \frac{3}{80}Mh^2\dot{\theta}^2cos^2\alpha + \frac{3}{10}Mb^2\left(-\frac{\dot{\theta}}{sin\alpha} + \dot{\theta}sin\alpha\right)^2\right]$
$T = T_{cm} + \frac{1}{2}M\dot{\theta}^2\left[\frac{3}{20}b^2cos^2\alpha + \frac{3}{80}h^2cos^2\alpha + \frac{3}{10}b^2\left(-\frac{cos^2\alpha}{sin\alpha}\right)^2\right]$
$T = T_{cm} + \frac{1}{2}M\dot{\theta}^2\left[\frac{3}{20}b^2cos^2\alpha + \frac{3}{80}h^2cos^2\alpha + \frac{3}{10}b^2\frac{cos^4\alpha}{sin^2\alpha}\right]$
$T = T_{cm} + \frac{1}{2}M\dot{\theta}^2\left[\frac{3}{20}b^2cos^2\alpha + \frac{3}{80}h^2cos^2\alpha + \frac{3}{10}b^2\frac{cos^4\alpha}{sin^2\alpha}\right]$
$T = T_{cm} + \frac{1}{2}M\theta^2\left[\frac{3}{20}b^2\left(cos^2\alpha + \frac{2cos^4\alpha}{sin^2\alpha}\right) + \frac{3}{80}h^2cos^2\alpha\right]$
substitutitute $b = h\tan\alpha$
$T = T_{cm} + \frac{1}{2}M\dot{\theta}^2\left[\frac{3}{20}h^2\left(sin^2\alpha + 2cos^2\alpha\right) + \frac{3}{80}h^2cos^2\alpha\right]$
$T = T_{cm} + \frac{1}{2}M\dot{\theta}^2\left[\frac{3}{20}h^2\left(1 + cos^2\alpha\right) + \frac{3}{80}h^2cos^2\alpha\right]$
$T = T_{cm} + \frac{1}{2}M\dot{\theta}^2h^2\left[\frac{3}{80}\left(4 + 5cos^2\alpha\right)\right]$
$s_{cm} = \frac{3}{4}h$ the coordinate of cm along the rotation axis through the CM
$v_{cm} = \frac{3}{4}\dot{\theta}hcos\alpha$
$T = \frac{1}{2}M\left(\frac{3h}{4}\dot{\theta}cos\alpha\right)^2 + \frac{1}{2}M\dot{\theta}^2h^2\left[\frac{3}{80}\left(4 + 5cos^2\alpha\right)\right]$
$T = \frac{1}{2}M\dot{\theta}^2h^2\left[\frac{9}{16}\cos^2\alpha + \frac{3}{80}\left(4 + 5cos^2\alpha\right)\right]$
$T = \frac{1}{2}M\dot{\theta}^2h^2\left[\frac{3}{20}\left(1 + 5cos^2\alpha\right)\right]$
$T = \frac{3}{40}M\dot{\theta}^2h^2\left(1 + 5cos^2\alpha\right)$
$M = \rho Vol = \frac{1}{3}\rho\pi h(htan\alpha)^2 = \frac{1}{3}\rho\pi h^3tan\alpha^2$
$T = \frac{1}{40}\dot{\theta}^2\rho\pi h^5tan^2\alpha\left(1 + 5cos^2\alpha\right)$
replace $\dot{\theta}^2 = \frac{4\pi^2}{\tau^2}$
$T = \frac{1}{10}\frac{\rho\pi^3h^5}{\tau^2}tan^2\alpha\left(1 + 5cos^2\alpha\right)$
the potential $U = Mgs_{cm}sin\alpha = \frac{3}{4}Mghsin\alpha$
where $s_{cm} = \frac{3}{4}h$ is being used

$L = T - U = \frac{3}{40} M \dot{\theta}^2 h^2 \left(1 + 5cos^2\alpha\right) - \frac{3}{4} Mghsin\alpha$
$\partial L_{\dot{\theta}} = \frac{3}{20} M \dot{\theta} h^2 \left(1 + 5cos^2\alpha\right)$
$\frac{d}{dt}\left(\partial L_{\dot{\theta}}\right) = \frac{3}{20} M \ddot{\theta} h^2 \left(1 + 5cos^2\alpha\right)$
$\partial L_{\theta} = 0$
$\frac{d}{dt}\left(\partial L_{\dot{\theta}}\right) - \partial L_{\theta} = 0$
by plugging in the values we find
$\frac{3}{20} M \ddot{\theta} h^2 \left(1 + 5cos^2\alpha\right) = 0$
$\dot{\theta} = const$

——————————————HW5 prob 4——————————————-

Assuming a reference frame with O centered in pivot(the fixed point) and Z normal to the paper as in the figure.

$\omega_z = \dot{\theta}$;$\omega_x = \omega_y = 0$

The kinetic energy is given by $T = \omega_k I_{kl} \omega_l / 2 = I\omega^2/2$

where $I = n_k I_{kl} n_l = I_{zz}$ and $\omega_k = |\omega| n_k$

$T = \frac{1}{2} I_{zz} \dot{\theta}^2$

$U = -Mglcos\theta$

$L = T - U = \frac{1}{2} I_{zz} \dot{\theta}^2 + Mglcos\theta$

$\partial L_{\dot{\theta}} = I_{zz} \dot{\theta}$

$\frac{d}{dt} \left(I_{zz} \dot{\theta} \right) = I_{zz} \ddot{\theta}$

$\partial L_{\theta} = Mglsin\theta$

$\frac{d}{dt} \left[\partial L_{\dot{\theta}} \right] - \partial L_{\theta} = 0 = I_{zz} \ddot{\theta} - Mglsin\theta$

$\ddot{\theta} = -Mglsin\theta / I_{zz}$

Alternative solution for prob 4

the torque $\overrightarrow{\tau} = \overrightarrow{R_{cm}} \times \overrightarrow{F}$

$\overrightarrow{F} = -mg \left(sin\theta \hat{i} + cos\theta \hat{j} \right)$

$\overrightarrow{R_{cm}} = -l\widehat{j}$

$$\overrightarrow{\tau} = \begin{bmatrix} i & j & k \\ 0 & -l & 0 \\ -mgsin\theta & -mgcos\theta & 0 \end{bmatrix} = -lMgsin\theta\widehat{k}$$

$\overrightarrow{\tau} = I \overrightarrow{\ddot{\theta}}$

hence $\ddot{\theta} = -Mglsin\theta / I_{zz}$

——————————HW5 prob 5——————————-

5. Let's follow the picture. Let us introduce an inertial coordinate system where the x-axis points straight down and the y-axis to the right. The point O where the rod attaches to the massive object moves with speed $v_O = r\dot{\theta}$. Thus its componts in our inertial coordinate system are

$$v_{Ox} = -r\dot{\theta}\sin\theta, \qquad v_{Oy} = r\dot{\theta}\cos\theta, \qquad v_{Oz} = 0.$$

The mass rotates about O with angular speed $\omega = \dot{\phi}$. [The ω contains $\dot{\phi}$ alone (and not also $\dot{\theta}$) because ϕ is measured with respect to the vertical (and not the rod), so that ϕ alone describes how much the object is rotated.] Thus its componts in our inertial coordinate system are

$$\omega_{Ox} = 0, \qquad \omega_{Oy} = 0, \qquad \omega_{Oz} = \dot{\phi}.$$

Let us denote the position vector of a mass point i with repsect to O by $\vec{r}_i$, and the center of mass position vector by $\vec{R}$. Then the kinetic enery is

$$T = \frac{1}{2}\sum_i m_i(\vec{v}_O + \vec{\omega}\times\vec{r}_i)^2 = \frac{M}{2}v_O^2 + M\vec{v}_O\cdot(\vec{\omega}\times\vec{R}) + \frac{1}{2}\vec{\omega}^T I_O\vec{\omega}$$

where I_O is the moment of inertia tensor for rotations about O. Now note that in our inertial coordinate we have

$$R_x = \bar{x}\cos\phi - \bar{y}\sin\phi, \qquad R_y = \bar{x}\sin\phi + \bar{y}\cos\phi, \qquad R_z = 0.$$

Thus $\vec{v}_O\cdot(\vec{\omega}\times\vec{R}) = r\dot{\theta}\dot{\phi}(\bar{x}\cos(\theta-\phi) + \bar{y}\sin(\theta-\phi))$ so that we obtain

$$T = \frac{M}{2}r^2\dot{\theta}^2 + Mr\dot{\theta}\dot{\phi}(\bar{x}\cos(\theta-\phi) + \bar{y}\sin(\theta-\phi)) + \frac{1}{2}I_{Ozz}\dot{\phi}^2$$

For the potential energy U we only need the hight of the CM above some zero level. We get

$$U = Mgh_{CM} + \text{const} = -Mg\left[r\cos\theta + l\cos(\phi+\alpha)\right],$$

where the constant l and α are $l = \sqrt{\bar{x}^2+\bar{y}^2}$, $\alpha = \arctan(\bar{y}/\bar{x})$. Since $l\cos(\phi+\alpha) = l\cos\alpha\cos\phi - l\sin\alpha\sin\phi)$ and $l\cos\alpha = \bar{x}$, $l\sin\alpha = \bar{y}$ we also have

$$U = -Mg\left[r\cos\theta + \bar{x}\cos\phi - \bar{y}\sin\phi\right].$$

[Note that we could replace the coordinate ϕ by $\gamma = \phi + \alpha$. This is equivalent to $\phi \to \gamma$, $\dot{\phi} \to \dot{\gamma}$, $\bar{y} \to 0$, $\alpha \to 0$, $\bar{x} \to l$ in both T and U.]
As always we have $L = T - U$ and thus obtain
$\partial L_{\dot{\theta}} = Mr^2\dot{\theta} + Mr\dot{\phi}\left[\bar{x}cos(\theta-\phi) + \bar{y}sin(\theta-\phi)\right]$
$\frac{d}{dt}(\partial L_{\dot{\theta}}) = Mr^2\ddot{\theta} - Mr\ddot{\phi}(\dot{\theta}-\dot{\phi})\left[\bar{x}sin(\theta-\phi) - \bar{y}cos(\theta-\phi)\right]$
$\partial L_{\theta} = -Mr\dot{\theta}\dot{\phi}\left[\bar{x}sin(\theta-\phi) - \bar{y}cos(\theta-\phi)\right] - Mgr\sin\theta$
$\partial L_{\dot{\phi}} = I_{zz}\dot{\phi} + Mr\dot{\theta}\left[\bar{x}cos(\theta-\phi) + \bar{y}sin(\theta-\phi)\right]$
$\frac{d}{dt}\left(\partial L_{\dot{\phi}}\right) = I_{zz}\ddot{\phi} - Mr\ddot{\theta}(\dot{\theta}-\dot{\phi})\left[\bar{x}sin(\theta-\phi) - \bar{y}cos(\theta-\phi)\right]$
$\partial L_{\phi} = Mr\dot{\theta}\dot{\phi}\left[\bar{x}sin(\theta-\phi) - \bar{y}cos(\theta-\phi)\right] - Mgl\sin(\phi+\alpha)$

eqn of motion are obtained by plugging the above values into the following
$\frac{d}{dt}(\partial L_{\dot{\theta}}) - \partial L_{\theta} = 0$
$\frac{d}{dt}\left(\partial L_{\dot{\phi}}\right) - \partial L_{\phi} = 0$

Alternative method

$$R_i = R_{CM} + r_i \qquad\qquad V_i = V_{CM} + v_i$$

r_i and v_i are position and velocity relative to CM R_{CM}. Always:

$$T = \frac{M}{2}V_{CM}^2 + \sum_i \frac{m_i}{2}v_i^2$$

Here: $r_i = D(\phi)r_i(t=0)$, i.e. r_i is a rotation around CM.
Thus

$$|v_i| = (\dot{\phi})|r_i(t=0)|$$

and

$$\sum_i \frac{m_i}{2}v_i^2 = \frac{I_{CM}}{2}(\dot{\phi})^2$$

Hence

$$T = \frac{M}{2}V_{CM}^2 + \frac{I_{CM}}{2}(\dot{\phi})^2$$

V_{CM} can be expressed in terms of θ and ϕ. This will lead to the same answer.