Mechanics - PHY 6247

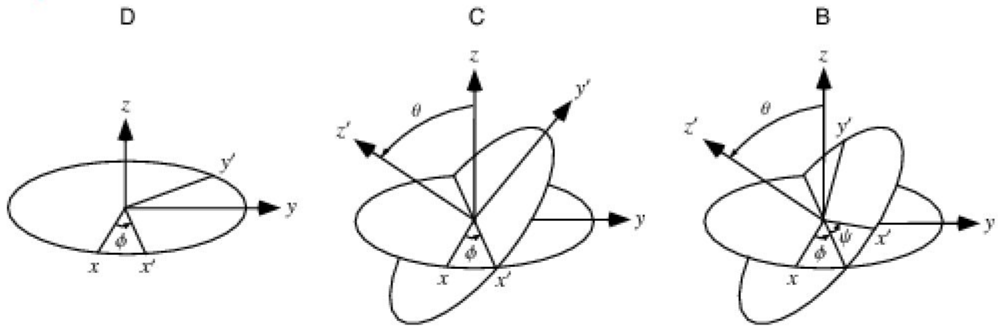
Solutions to HW 4

====================Prob 1=====================
orthogonality cond: $A^T A = I = AA^T$
$A^T = A^{-1}$
Let A, B ortogonal
$AA^T = A^T A = I = BB^T = B^T B$
recall: $(XY)^T = Y^T X^T$
$(AB)(AB)^T = ABB^T A^T = AIA^T = AA^T = I$
The above proves that
the product of two orthogonal matrices is ortogonal itself.

====================Prob 2=====================
$X' = BXB^{-1}$ similarity transformation for the matrix X
$Tr(X') = Tr(BAB^{-1})$
recall $Tr(AB) = Tr(BA) \Rightarrow$
$Tr[B(XB^{-1})] = Tr[(XB^{-1})B] = Tr(X)$
this proves that $Tr(BXB^{-1}) = Tr(X)$
or the trace of a matrix is invariant under similarity transformation

====================Prob 3=====================
a matrix A is antisimmetric or skew symmetric if $A = -A^T$
Let's assume $A' = BAB^{-1}$ where B is orthogonal $BB^T = I$. Then
$(A')^T = (BAB^{-1})^T = (B^{-1})^T A^T B^T = BA^T B^T = -BAB^T = -A'$.
Or using components, a matrix A is antisimmetric if
$a_{ij} = -a_{ji}$
prove that antysimmetry is preserved under a similarity transf
Let's assume $A' = BAB^{-1}$ where B is orthogonal $BB^T = I$ or
$(bab^{-1})_{ij} = b_{ik} a_{kl} b^{-1}_{lj}$
$b^{-1}_{lj} = b_{jl}$
$(bab^{-1})_{ij} = b^{-1}_{ki}(-a_{lk}) b_{jl} = -b^{-1}_{ki} a_{lk} b_{jl} = -b_{jl} a_{lk} b^{-1}_{ki}$
$(bab^{-1})_{ij} = -(bab^{-1})_{ji}$
which is the antysimmetric property is invariant under
similarity orthogonal transformation.

====================Prob 4====================



According to Euler's rotation theorem, any rotation may be described using three angles. If the rotations are written in terms of rotation matrices D, C, and B, then a general rotation A can be written as

$$A = B\,C\,D. \qquad (1)$$

we seek all the projection of $\dot{\phi} = \omega_\phi, \dot{\theta} = \omega_\theta, \dot{\psi} = \omega_\psi$onto x,y,z
the so called space frame.

$$\begin{array}{lllllll} \omega_x & = & \dot{\phi}_x & + & \dot{\theta}_x & + & \dot{\psi}_x \\ \omega_y & = & \dot{\phi}_y & + & \dot{\theta}_y & + & \dot{\psi}_y \quad (1) \\ \omega_z & = & \dot{\phi}_z & + & \dot{\theta}_z & + & \dot{\psi}_z \end{array}$$

It is possible to see the projections from the picture.

ω_ϕ *is* parallel to z hence$\omega_\phi = \dot{\phi}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

ω_θ is parallel to the line of nodes and orthogonal to z

hence$\omega_\theta = \dot{\theta}\begin{bmatrix} cos\phi \\ sin\phi \\ 0 \end{bmatrix}$

$\omega_\psi$is parallel to z'hence $\omega_\psi = \dot{\psi}\begin{bmatrix} sin\theta sin\phi \\ -sin\theta cos\phi \\ cos\theta \end{bmatrix}$

now we can fill out (1) with the found projection.

$\omega_x = 0 + \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi$

$\omega_y = 0 + \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi$

$\omega_z = \dot{\phi} + \dot{\psi}\cos\theta$

====================Prob 5=================

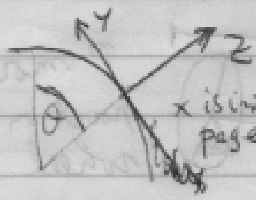
HW

~~???~~ throw up ball on earth. Consider Coriolis force.

in frame of earth

$$m\vec{a} = m\vec{g} - 2m(\vec{\omega}\times\vec{v}) - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

assume $m\vec{g} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \vec{F} \simeq m\vec{g}$

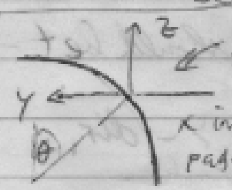


$$\vec{V}_0 = \begin{pmatrix}0\\0\\V_0\end{pmatrix} \qquad m\vec{g} = -\begin{pmatrix}0\\0\\mg\end{pmatrix}$$

$$\vec{\omega} = \begin{pmatrix}0\\\omega_y\\\omega_z\end{pmatrix} \qquad \omega_z = \omega\cos\theta,\ \omega_y = \omega\sin\theta$$

$$\begin{cases} m\ddot{x} = -2m(\omega_y V_z - \omega_z V_y) = \\ m\ddot{y} = -2m(\omega_z V_x - 0) \\ m\ddot{z} = \cancel{\ldots} - mg - 2m(0 - \omega_y V_x) \end{cases}$$

Not used



$$\vec{V}_0 = \begin{pmatrix}0\\v_0\cos\theta\\-v_0\sin\theta\end{pmatrix} \qquad m\vec{g} = -mg\begin{pmatrix}0\\-\sin\theta\\\cos\theta\end{pmatrix}$$

$$\vec{\omega} = \begin{pmatrix}0\\0\\\omega\end{pmatrix}$$

$$\left.\begin{aligned} m\ddot{x} &= -2m(0 - \omega V_y) \\ m\ddot{y} &= +mg\sin\theta - 2m(\omega V_x - 0) \\ m\ddot{z} &= -mg\cos\theta - 2m\cdot 0 \end{aligned}\right\} \text{ but } \theta = \theta(x,y,z)$$

$\rightarrow g \sim 10\frac{m}{s^2}$, $\omega \simeq \frac{2\pi}{24h} \sim \frac{1}{4h} \sim 10^{-5}/s$, $V_{max} = V_0 \sim 100\frac{m}{s}$ $\Rightarrow V_{max}\cdot\omega \sim 10^{-3}\frac{m}{s^2}$

$\Rightarrow$ g completely dominates $\omega_y V_x$ term in $\ddot{z} = -g + 2\omega_y V_x \sim -g$

$\Rightarrow \dot{z} \sim V_0 - gt \quad \Rightarrow$ time at z_{max}: $t_h \simeq \frac{V_0}{g} \sim 10s$

initially: $\ddot{x} \sim \omega V_0 \sim 10^{-3}\frac{m}{s^2} \Rightarrow V_{x,max} < 10^{-3}\frac{m}{s^2}\cdot 10s \sim 10^{-2}\frac{m}{s} \ll V_0$

$\ddot{y} \lesssim \omega V_{x,max} \sim 10^{-5}\frac{1}{s}\,10^{-2}\frac{m}{s} \sim 10^{-7}\frac{m}{s^2} \Rightarrow V_{y,max} < 10^{-7}\frac{m}{s^2}\cdot 10s \sim 10^{-6}\frac{m}{s} \Rightarrow V_y \ll V_x$

$$\text{so}\quad \begin{aligned} \ddot{x} &= -2(\omega_y V_z - \omega_z V_y) &&\approx -2\omega_y V_z \\ \ddot{y} &= -2\omega_z V_x &&\approx 0 \\ \ddot{z} &= -g + 2\omega_y V_x &&\approx -g \end{aligned}$$

1st particle

from ground: $\underline{V_{z_1} = V_0 - gt} \Rightarrow z_1 = V_0 t - \frac{1}{2} g t^2$

back at ground: $z_1(T) = 0 \Rightarrow 0 = V_0 T - \frac{1}{2} g T^2$

$$\Rightarrow T = \frac{2V_0}{g}$$

$$\ddot{x}_1 = -2\omega_y (V_0 - gt)$$

$$\dot{x}_1 = -2\omega_y \left(V_0 t - \frac{1}{2} g t^2\right) + 0 \longleftarrow \dot{x}(0) = 0$$

$$x_1 = -2\omega_y \left(\frac{V_0}{2} t^2 - \frac{1}{6} g t^3\right) + 0 \longleftarrow x(0) = 0$$

$$\underline{x_1(T)} = -2\omega_y \left(\frac{V_0}{2} 4 \frac{V_0^2}{g^2} - \frac{g}{6} 8 \frac{V_0^3}{g^3}\right) = \underline{-2\omega_y \left(\frac{2}{3}\right) \frac{V_0^3}{g^2}}$$

2nd part.

from height where V_{z_1} of 1st part. was 0 $\Rightarrow t_2 = \frac{V_0}{g}$

$$\Rightarrow z_1(t_2) = V_0 \frac{V_0}{g} - \frac{1}{2} g \frac{V_0^2}{g^2} = \underline{\frac{1}{2} \frac{V_0^2}{g}}$$

time to fall down is $t_2 = \frac{V_0}{g}$;

$\cancel{\ddot{x} = -2\omega_y(-gt)}$ and $\underline{V_{z_2} = -gt}$

$$\ddot{x}_2 = -2\omega_y (-gt)$$

$$\dot{x}_2 = -2\omega_y \left(-\frac{1}{2} g t^2\right) + 0$$

$$x_2 = -2\omega_y \left(-\frac{1}{6} g t^3\right) + 0$$

$$x_2(t_2) = -2\omega_y \left(-\frac{1}{6} g \frac{V_0^3}{g^3}\right) = -2\omega_y \left(-\frac{1}{6}\right) \frac{V_0^3}{g^2}$$

$$\Rightarrow \frac{x_2(t_2)}{x_1(T)} = \frac{-1/6}{2/3} = -\frac{1}{4}$$

====================Prob 6====================

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}; M_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}; M_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_1M_2 - M_2M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$