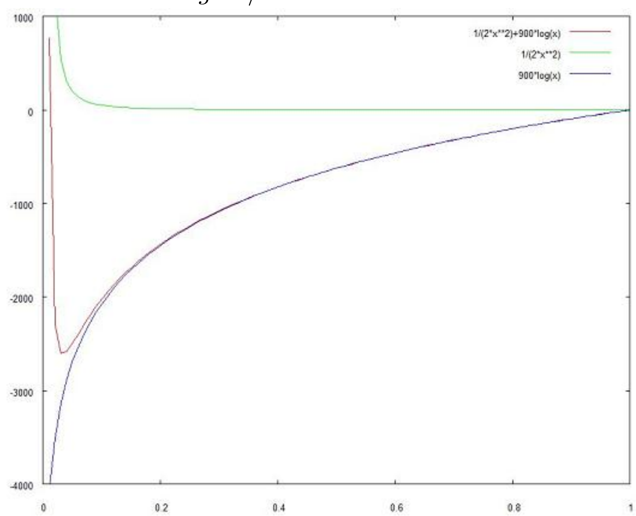
========================= PROB 1==================================

$V(r) = k ln(r/L)$

$k = 900J; L = 1m$

angular momentum is $l = 1kgm^2/s$



a) the plot

b) the minimim of the fictitious (effective) potential is when the motion is on a circular orbit.

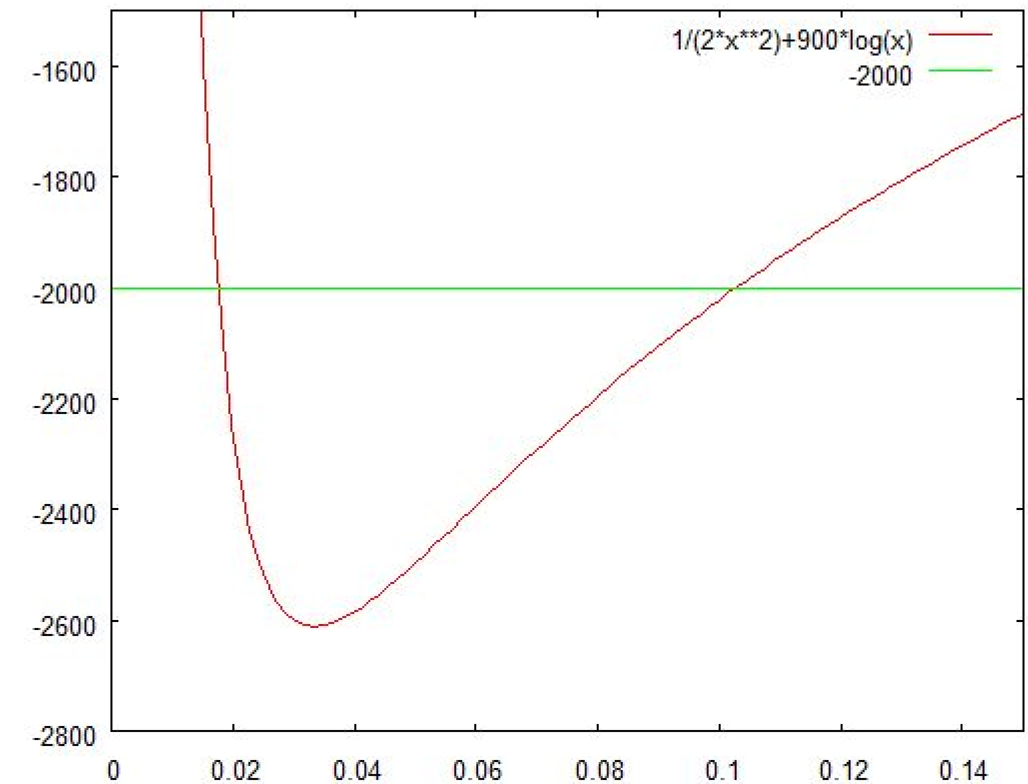
the fictitious potential is minimum when $-\frac{\partial V^{'}(r)}{\partial r} = 0 = f(r) + \frac{l^2}{mr^3} = -\frac{\partial V(r)}{\partial r} + \frac{l^2}{mr^3} = 0$ see Goldstein p77

$-\frac{\partial V(r)}{\partial r} = \frac{-k}{r} = \frac{-l^2}{mr^3}$

assuming $l = 1kgm^2/s; k = 900J; m = 1kg$

$r = \frac{l}{\sqrt{mk}} = \frac{1}{30}m$

c) absidal points are at the intersection of the energy with the fictitious potential



$E = V^{'}(r) + \frac{1}{2}m\dot{r}^2 = kln(r/L) + \frac{1}{2}\frac{l^2}{mr^2} + \frac{1}{2}m\dot{r}^2$
at the absidal points $E = V^{'}(r)$ for it is $\dot{r}^2 = 0$
hence by solving this eq $kln(r/L) + \frac{1}{2}\frac{l^2}{mr^2} = E$
we find the wanted values of r.
$r1 = 0.01743m$
$r2 = 0.10282m$
$\alpha = \frac{r1+r2}{2} = 0.06013$
$e = \frac{r2-r2}{r2+r1} = 0.71$

======================= PROB 2==============================

The energy of the planet before the impact is $E_i = \frac{1}{2}Mv_i^2 - \frac{k}{r}$ where v_i is the planet's initial speed, $r = r_{max}$ and $k = GM_{\odot}M$.

$E_f = \frac{1}{2}(M+m)v_f^2 - \frac{(M+m)k}{Mr}$ is the energy of the system (planet+comet) after the impact

Since this system is on parabolic orbit $E_f = 0 \Longrightarrow v_f^2 = \frac{2k}{Mr}$ (1)

From the conservation of momentum we get $(M+m)v_f = Mv_i + mv_c$, where v_c is the speed of the comet before impact.

Hence $mv_c = (M+m)v_f - Mv_i$ (2).

Next we need to find v_i. Recall that the planet's orbit before impact is given by $r(\theta) = d/(1 + e\cos\theta)$, where the semi-latus rectum $d = \frac{l^2}{Mk}$. (3)

For $\theta = \pi$ we obtain the maxium distance $r = d/(1-e)$, and thus $d = (1-e)r$ (4).

Using (3) and (4) we see $l^2 = (1-e)rMk$ (5).

Yet l is also $l = rMv_i$, which we insert in (5) to obtain $M^2 v_i^2 = (1-e)Mk/r = (1-e)M^2 \frac{k}{Mr}$ (6).

Using (2), the kinetic energy of the comet before impact is $T_c = \frac{1}{2m}(mv_c)^2 = \frac{1}{2m}[(M+m)^2 v_f^2 + M^2 v_i^2 - 2(M+m)v_f M v_i] = \frac{1}{2m}[(M+m)^2 \frac{2k}{Mr} + (1-e)M^2 \frac{k}{Mr} - 2(M+m)M\sqrt{\frac{2k}{Mr}(1-e)\frac{k}{Mr}}] = \frac{1}{2m}[2(M+m)^2 + (1-e)M^2 - 2(M+m)M\sqrt{2(1-e)}]\frac{k}{Mr} = \frac{k}{2mMr}[2(M+m)^2 - 2\sqrt{2(1-e)}(M+m)M + (1-e)M^2]$

The comet's total energy before impact is $E_c = T_c - \frac{mk}{Mr} = T_c - \frac{k}{2mMr}2m^2$. Since $m \ll M$ we see that the second term is much smaller.

In fact if we let $m/M \to 0$ we find

$$E_c = T_c = \frac{k}{2mMr}M^2[2-2\sqrt{2(1-e)}+(1-e)] = \frac{kM}{2mr}[2-2\sqrt{2(1-e)}+(1-e)]$$

Using $r = (1+e)a$ and $e = 1-\alpha$ we find

$$E_c = T_c = \frac{kM}{2m(2-\alpha)a}[2 - 2\sqrt{2\alpha} + \alpha].$$

Note: We had $T_c = \frac{kM}{2mr}[2 - 2\sqrt{2(1-e)} + (1-e)]$. If we set $r = d/(1-e)$, this becomes $T_c = \frac{kM}{2md}(1-e)(2 - 2\sqrt{2(1-e)} + (1-e))$. Since d is finite for $e \to 1$, we see that $T_c \to 0$ if $e \to 1$.

Note2: We could have gotten v_i also from $E_i = -\frac{k}{2a} = -(1+e)\frac{k}{2r}$ and the obvious $E_i = \frac{1}{2}Mv_i^2 - \frac{k}{r}$. We get again (6). Here the $E_i = -\frac{k}{2a}$ comes from adding the r_{min} and r_{max} found from the quadratic Eqn. $E_i = \frac{M}{2}(0 + r^2\left(\frac{l}{mr^2}\right)^2) - \frac{k}{r}$ that gives us r_{min} and r_{max}.

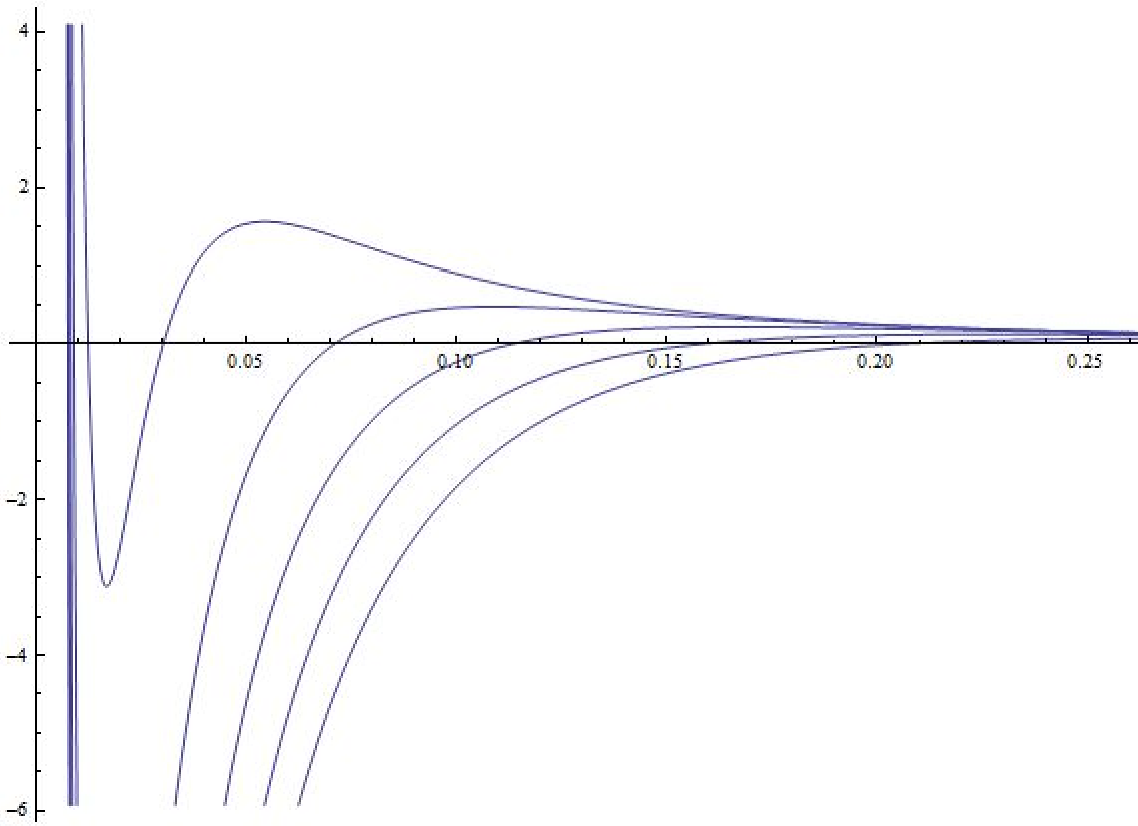
========================PROB 3===============================

```
In[78]:= k = 1.5
         l = 0.1
         Plot[Table[l^2/r^2 - k*e^(-r/a)/r, {a, 0.02, 0.06, .01}], {r, 0, .3}]

Out[78]= 1.5

Out[79]= 0.1

Out[80]=
```



The plot shows the effective potential for different values of a $0.02 \leq a \leq 0.06$.

Notice how at low values of $a = 0.02$ a maximum starts to showup.

The indicated values k, l, a, r are all fictitious.

In nuclear physics a is the range of the nuclear force: $\sim 10^{-15} m$

$V(r) = -\frac{k}{r} e^{\frac{r}{a}}$

$L = T - V$

$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r} e^{-\frac{r}{a}}$

$\frac{d}{dt}\left(\partial L_{\dot{\theta}}\right) - \partial L_{\theta} = 0$

$\partial L_{\dot{\theta}} = mr^2\dot{\theta}; \partial L_{\theta} = 0$

hence $\frac{d}{dt}\left(\partial L_{\dot{\theta}}\right) = 0$
and $mr^2\dot{\theta} = const$
(1) $l = mr^2\dot{\theta}$angular momentum of the sytem
$\frac{d}{dt}\left(\partial L_{\dot{r}}\right) - \partial L_r = 0$
$\partial L_{\dot{r}} = m\dot{r}$;$\partial L_r = mr\dot{\theta}^2 - \frac{k}{ar}e^{-\frac{r}{a}} - \frac{k}{r^2}e^{-\frac{r}{a}}$
$\frac{d}{dt}\left(\partial L_{\dot{r}}\right) = m\ddot{r}$
(2) $m\ddot{r} - mr\dot{\theta}^2 + \left(\frac{r}{a}+1\right)\frac{k}{r^2}e^{-\frac{r}{a}} = 0$
(1) and (2) are the equations of motion
we can plug (1) into (2)
(3) $m\ddot{r} - \frac{l^2}{mr^3} + \left(\frac{r}{a}+1\right)\frac{k}{r^2}e^{-\frac{r}{a}} = 0$
now that we know from (1) that l is a constant of the motion
so we can write the fictitious(effective) potential as
$V^{'}(r) = \frac{l^2}{2mr^2} + V(r)$
$V^{'}(r) = \frac{l^2}{2mr^2} - \frac{k}{r}e^{-\frac{r}{a}}$
$E = \frac{1}{2}m\dot{r}^2 + V^{'}(r)$
$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{k}{r}e^{-\frac{r}{a}}$
——————————————————- prob 3part b)
use $u = 1/r$ and show that $\frac{d^2u}{d\theta^2} + u = F(u) = -(m/l^2)\frac{d[V(1/u)]}{du}$
$du/dr = -1/r^2$;
$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\dot{r}\frac{1}{\dot{\theta}}$
hence from (1) $\frac{l}{mr^2} = \dot{\theta}$
$\frac{du}{d\theta} = -\frac{1}{r^2}\dot{r}\frac{mr^2}{l} = -\frac{m}{l}\dot{r}$
$\frac{d^2u}{d\theta^2} = \frac{dt}{d\theta}\frac{d}{dt}\left(-\frac{m}{l}\dot{r}\right) = -\frac{m}{l}\frac{1}{\dot{\theta}}\ddot{r} = -\frac{m}{l}\frac{m\ddot{r}^2}{l}\ddot{r}$
$\frac{d^2u}{d\theta^2} = -\frac{m^2r^2}{l^2}\ddot{r}$
$\ddot{r} = -\frac{d^2u}{d\theta^2}\frac{l^2}{m^2r^2} = -\frac{d^2u}{d\theta^2}\frac{l^2}{m^2}u^2$ (6)
replacing (6) into (3) $m\ddot{r} - \frac{l^2}{mr^3} + \left(\frac{r}{a}+1\right)\frac{k}{r^2}e^{-\frac{r}{a}} = 0$
$-\frac{d^2u}{d\theta^2}\frac{l^2}{m}u^2$-$\frac{l^2}{m}u^3 + \frac{d}{dr}V(r) = 0$ where we replaced $\frac{d}{dr}V(r) = \left(\frac{r}{a}+1\right)\frac{k}{r^2}e^{-\frac{r}{a}}$
$-\frac{d^2u}{d\theta^2}\frac{l^2}{m}u^2$-$\frac{l^2}{m}u^3 - u^2\frac{d}{du}V(u) = 0$
$\frac{d^2u}{d\theta^2}+u = -\frac{m}{l^2}\frac{d}{du}V(u) = F(u)$ which proves (i)
(ii) and (iii) Taylor expand..$u = u_0 + \delta u$; $F(u_0) + F^{'}(u_0)\delta u$
plug the above into the eq of motion $\frac{d^2u}{d\theta^2}+u = F(u)$ (7)
assume $u^{"} = \frac{d^2u}{d\theta^2}$; $F^{'} = \frac{d}{du}F$
$u_0^{"} + \delta^{"}u + u_0 + \delta u = F(u_0) + F^{'}(u_0)\delta u$
$u_0^{"} = 0$; for $u_0 = const$
from EOM (7) $u_0^{"} + u_0 = F(u_0) \Rightarrow 0 + u_0 = F(u_0)$(8)
$\delta u^{"} + [1 - F^{'}(u_0)]\delta u = F(u_0) - u_0 = 0$
$\delta u^{"} = -[1 - F^{'}(u_0)]\delta u$
assuming $\beta^2 = [1 - F^{'}(u_0)]$ (9)
$\frac{d^2u}{d\theta^2} = -\beta^2 u$
$\delta^{"}u = -\beta^2\delta u$
the solution is $\delta u = Acos\beta\delta\theta$

in order to find β we can use (i)

$V(u) = -kue^{\frac{-1}{au}}$

$F(u) = -\frac{m}{l^2}\frac{d}{du}V(u) = \left(-\frac{m}{l^2}\right)\left[-k - \left(\frac{1}{au^2}\right)\frac{ku}{a}\right]e^{\frac{-1}{au}}$

$F(u) = \frac{mk}{l^2}\left(1+\frac{1}{au}\right)e^{\frac{-1}{au}}$ (10)

now recall (8) $F(u_0) = u_0$

$F(u_0) = \frac{mk}{l^2}\left(1+\frac{1}{au_0}\right)e^{\frac{-1}{au_0}} = u_0$

$e^{\frac{-1}{au_0}} = u_0\frac{l^2}{mk}\left(\frac{1}{1+\frac{1}{au_0}}\right)$(11)

$F'(u) = \frac{mk}{l^2}e^{\frac{-1}{au}}\left\{\left[\frac{-1}{au^2}\right] + \left[1+\frac{1}{au}\right]\left(\frac{1}{au^2}\right)\right\}$

$F'(u) = \frac{mk}{l^2}e^{\frac{-1}{au}}\left(\frac{1}{au^3}\right) = \frac{mk}{a^2u^3l^2}e^{\frac{-1}{au}}$(12)

$F'(u_0) = \frac{mk}{a^2u_0^3l^2}e^{\frac{-1}{au_0}}$ (13)

plug (11) into (13)

$F'(u_0) = \frac{mk}{au_0^3l^2}\frac{u_0l^2}{mk}\left(\frac{1}{1+\frac{1}{au_0}}\right)$

$F'(u_0) = \frac{1}{a^2u_0^2}\left(\frac{au_0}{1+au_0}\right) = \frac{1}{au_0(1+au_0)}$

$\beta^2 = [1 - F'(u_0)] = 1 - \frac{1}{au_0(1+au_0)}$

For $r_0/a \ll 1$ or $au_0 \gg 1$ we find $\beta^2 \approx 1 - \frac{1}{(au_0)^2} = 1 - (r_0/a)^2$ and thus $\beta \approx 1 - \frac{1}{2}(r_0/a)^2$.

From $\beta\Delta\theta = 2\pi$ we obtain $\Delta\theta = 2\pi/\beta \approx 2\pi + \pi(r_0/a)^2$.

======================PROB 4================================

given lunar month $T_{moon} = 27.3$days ; earth period about the sun $T_e = 365.4$ days

mean radii of the earth's orbit about the sun $r_{earth} = 1.49 \times 10^{11}m$

mean radii of the moon's orbit about the earth $r_{moon} = 3.8 \times 10^8 m$

from Kepler III law the period of a planet is $T = 2\pi a^{3/2}\sqrt{\mu/k}$

assuming $\mu_{es} = \frac{m_e \times M_s}{m_e + M_s} \approx M_e$ the reduced mass of the sun , earth system

similarly assuming $\mu_{me} = \frac{m_m \times M_e}{m_m + M_e} \approx m_m$ the reduced mass of the earth , moon system

$k = GMm$

a_{earth} earth's orbit semiaxis major about the sun

a_{moon}moon's orbit semiaxis major about the earth

$T_e^2 = \frac{4\pi^2 a_e^3}{GM_s}$, similarly $T_m^2 = \frac{4\pi a_m^3}{GM_e}$

$\frac{T_m^2}{T_e^2} = \left(\frac{a_m}{a_e}\right)^3\frac{M_s}{M_e}$

$\frac{M_s}{M_e} = \left(\frac{a_e}{a_m}\right)^3\left(\frac{T_m}{T_e}\right)^2$

plug in numbers to find that

$\frac{M_s}{M_e} = \left(\frac{1.49\times10^8}{3.8\times10^5}\right)^3\left(\frac{27.3}{365.4}\right)^2 = 3.365 \times 10^5$

======================PROB 5================================

$f(r) = -\frac{\partial V}{\partial r}$

$\frac{1}{r^2}\frac{d}{d\theta}\left(\frac{l}{Mr^2}\frac{dr}{d\theta}\right) - \frac{l^2}{Mr^3} = f(r)$ (Goldstein 3.33)

given $r = c\theta^2 \Rightarrow \frac{dr}{d\theta} = 2c\theta$
$\frac{1}{r^2}\frac{d}{d\theta}(\frac{l}{Mr^2}2c\theta) - \frac{l^2}{Mr^3} = f(r)$
substitute $r^2 = c^2\theta^4$
$\frac{1}{r^2}\frac{d}{d\theta}(\frac{2lc\theta}{Mc^2\theta^4}) - \frac{l^2}{Mr^3} = f(r)$
$\frac{1}{r^2}\frac{d}{d\theta}(\frac{-2l}{Mc\theta^3}) - \frac{l^2}{Mr^3} = f(r)$
$\frac{1}{r^2}(\frac{-6l}{Mc\theta^4}) - \frac{l^2}{Mr^3} = f(r)$
$\frac{1}{r^2}(\frac{-6l}{Mc\theta^4}) - \frac{l^2}{r^3} = f(r)$
replace $r = c\theta^2$
$\frac{-6cl}{Mr^4} - \frac{l^2}{Mr^3} = f(r)$
$V(r) = Ar^n + Br^m$
$-\frac{\partial V}{\partial r} = -Anr^{n-1} - Bmr^{m-1} = f(r) = \frac{-6cl}{M}r^{-4} - \frac{l^2}{M}r^{-3}$
hence $n = -3; m = -2$