==============PROBLEM 1===============

We can solve this in at least 2 ways. We first present a way where we adapt the coordinate system to simplify the problem:

*Method 1:

In spherical coordinates on the surface of the unit sphere, we have

$ds^2 = d\theta^2 + sin^2\theta d\phi^2$

$S = \int_1^2 ds = \int \sqrt{1 + sin^2\theta\phi'^2} d\theta$

$\phi' = \frac{d\phi}{d\theta}$

The Euler conditions to minimize S

$\frac{d}{d\theta}\left(\frac{\partial F}{\partial\phi'}\right) - \frac{\partial F}{\partial\phi} = 0$

in this case

$F = \sqrt{1 + sin^2\theta\phi'^2}$

$\frac{\partial F}{\partial\phi'} = \frac{\phi' sin^2\theta}{\sqrt{1+sin^2\theta\phi'^2}}$

$\frac{\partial F}{\partial\phi} = 0$

hence $\frac{d}{d\theta}\left(\frac{\partial F}{\partial\phi'}\right) = 0$ and $\frac{\partial F}{\partial\phi'} = k = constant$

$\frac{\phi' sin^2\theta}{\sqrt{1+sin^2\theta\phi'^2}} = k$

We have to solve this equation, starting at θ_0 with some value for ϕ and ϕ' . To simplify the situation we rotate our cordinate system such that $\phi' = 0$ at θ_0 . Then it follows that $k = 0$ and the differential equation simplifies to

$\frac{\phi' sin^2\theta}{\sqrt{1+sin^2\theta\phi'^2}} = 0$

which yields

$\phi' = 0.$

We can integrate the latter and obtain $\phi = const$, which is obviously a great circle.

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*Method 2:

In spherical coordinates on the surface of a sphere of radis R, we have

$ds^2 = R^2 d\theta^2 + R^2 sin^2\theta d\phi^2$

$S = \int_1^2 ds = R\int \sqrt{1 + sin^2\theta\phi'^2} d\theta$

$\phi' = \frac{d\phi}{d\theta}$

The Euler conditions to minimize S

$\frac{d}{d\theta}\left(\frac{\partial F}{\partial\phi'}\right) - \frac{\partial F}{\partial\phi} = 0$

in this case

$F = R\sqrt{1 + sin^2\theta\phi'^2}$

$\frac{\partial F}{\partial\phi'} = \frac{R\phi' sin^2\theta}{\sqrt{1+sin^2\theta\phi'^2}}$

$\frac{\partial F}{\partial\phi} = 0$

hence $\frac{d}{d\theta}\left(\frac{\partial F}{\partial\phi'}\right) = 0$ and $\frac{\partial F}{\partial\phi'} = constant$

$\frac{\phi' sin^2\theta}{\sqrt{1+sin^2\theta\phi'^2}} = k$ where R is factorized in the constant k

by squaring the former expression we get

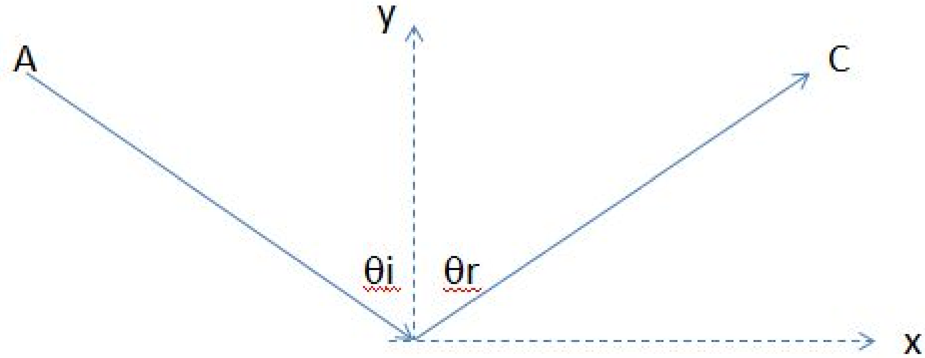
$\phi'^2 sin^4\theta = k^2(1 + sin^2\theta\phi'^2)$
$\phi'^2(sin^4\theta - k^2 sin^2\theta) = k^2$
$\phi'^2 = \frac{k^2}{sin^4(\theta)(1-k^2\frac{1}{sin^2\theta})}$
$d\phi = \frac{k\times csc^2\theta}{\sqrt{1-k^2csc^2\theta}}d\theta$
Let's apply the following substitution:
$1 + cot^2\theta = csc^2\theta$
$\phi = \int \frac{kcsc^2\theta d\theta}{k\sqrt{\frac{1}{k^2}-(1+cot^2\theta)}} \Rightarrow \int \frac{csc^2\theta d\theta}{\sqrt{\frac{1}{k^2}-1-cot^2\theta)}}$
let's assign $a^2 = \frac{k^2}{1-k^2}$
$\phi = \int \frac{csc^2\theta d\theta}{\sqrt{\frac{1}{a^2}-cot^2\theta)}} = a\int \frac{csc^2\theta d\theta}{\sqrt{1-a^2cot^2\theta)}}$
when we assign
(1) $x = cot\theta \Rightarrow dx = -csc^2\theta d\theta$
$\phi = a\int \frac{-dx}{\sqrt{1-a^2x^2}}$
the solution to the above integral is $\phi(x) = -sin^{-1}(ax) + \beta$
we then transform back to θ by pluggin in $x = cot\theta$ from 1)
$\phi(\theta) = -sin^{-1}(acot\theta) + \beta \Rightarrow sin(\beta - \phi) = acot\theta$
$cot\theta = \frac{1}{a}(sin\beta cos\phi - sin\phi cos\beta)$
assign $c1 = \frac{1}{a}sin\beta; c2 = \frac{1}{a}cos\beta$
$cot\theta = c1cos\phi - c2sin\phi$
let's multiply by $rsin\theta \Longrightarrow rcos\theta = r(c1sin\theta cos\phi - c2sin\theta sin\phi)$
which in spherical coordinates
$x = rsin\theta cos\phi; y = rsin\theta sin\phi; z = rcos\theta$
is a plane through the center of the coordinate axis.
$Ax + By + cz = 0$
In conclusion by applying the Euler eqn we've found that the condition for S to be an extremum (supposedly a minimum) is that the function F satisfies the eqn of a plane through the center of a sphere. Hence the shortest path is always a great circle through the points A,B produced by the intersection of such a plane with a sphere of radius R centered at the origin.

==============PROBLEM 2===============

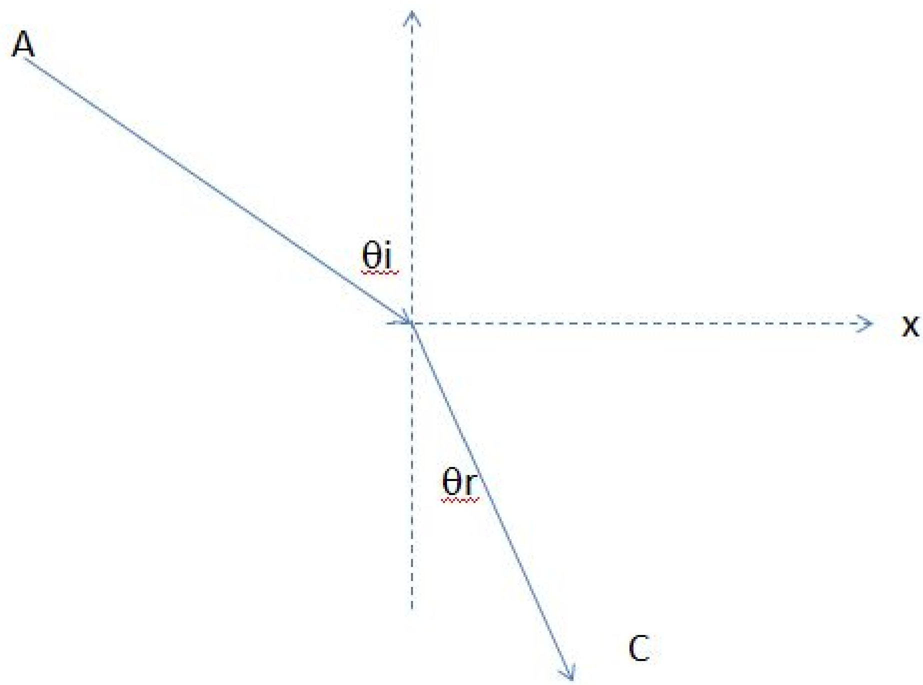
the action is $S = \int Ldt$ where L is the Lagrangian. Let's define the constant velocities
$v_0 = \frac{x-x_0}{t-t_0}$
$v_1 = \frac{x_1-x}{t_1-t}$
part a)
The Lagrangian is
$L = \frac{1}{2}mv_0^2$ if the time is between t_0 and t
$L = \frac{1}{2}mv_1^2$ if the time is between t and t_1
$S = \int Ldt = \frac{1}{2}mv_0^2(t-t_0) + \frac{1}{2}mv_1^2(t_1-t) = \frac{1}{2}m\left[\frac{(x-x_0)}{t-t_0}^2 + \frac{(x_1-x)}{t_1-t}^2\right]$
part b)
in order to minimize the action:

$\frac{dS}{dx} = 0 \Rightarrow m\left[\frac{(x-x_0)}{t-t_0} - \frac{(x_1-x)}{t_1-t}\right] = 0$
hence $v_1 = v_0$

==============PROBLEM 3===============



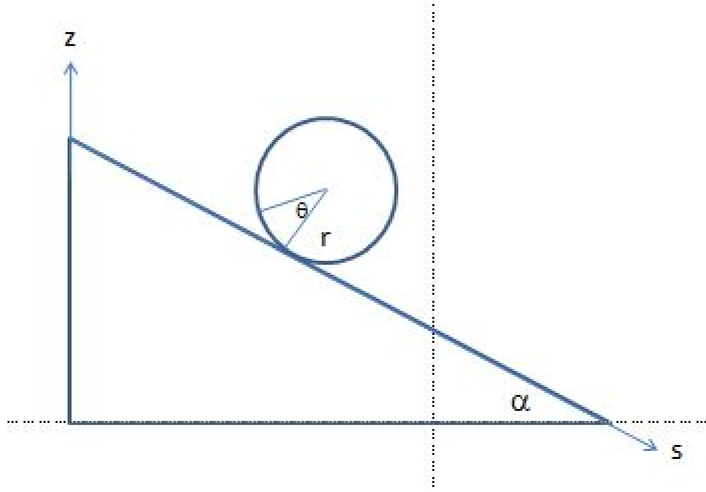
part 3a)
Assuming c is the speed of light in a vacuum
$t_{AB} = \frac{\sqrt{(x-x_A)^2+y_A^2}}{c}$ is the travel time from A to B
$t_{BC} = \frac{\sqrt{(x_C-x)^2+y_C^2}}{c}$ is the travel time from B to C
according to Fermat the travel time is a minimum for light
$\frac{dt}{dx} = 0 \Rightarrow \frac{dt}{dx} = \frac{d}{dx}(t_{AB} + t_{BC}); = \frac{1}{c}\left[\frac{x-x_A}{\sqrt{(x-x_A)^2+y_A^2}} - \frac{x-x_C}{\sqrt{(x_C-x)^2+y_C^2}}\right] = 0$
$sin(\theta_i) = \frac{x-x_A}{\sqrt{(x-x_A)^2+y_A^2}}$
$sin(\theta_r) = \frac{x-x_c}{\sqrt{(x-x_c)^2+y_c^2}}$
from which it is clear that $\theta_i = \theta_r$.
part 3b)



$t_{AB} = \frac{\sqrt{(x-x_A)^2+y_A^2}}{c/n_i}$ is the travel time from A to B in the medium i
$t_{BC} = \frac{\sqrt{(x_C-x)^2+y_C^2}}{c/n_r}$ is the travel time from B to C in the mediun r
$\frac{dt}{dx} = 0$ Fermat principle
$\left[\frac{(x-x_A)n_i}{\sqrt{(x-x_A)^2+y_A^2}} - \frac{(x-x_C)n_r}{\sqrt{(x_C-x)^2+y_C^2}}\right] = 0$
$n_i sin\theta_i = n_r sin\theta_r$
which is the Snell's law
==============PROBLEM 4===============
The Lagrangian of the particle moving in the uniform gravitational field is
$L = \frac{1}{2}m\left(\frac{dz}{dt}\right)^2 - mgz$
$z(t) = z_0 + v_0 t + \frac{1}{2}at^2$ and
(1) $z = z_0$when $t = 0$;
(2) $z = z_1$when $t = t_1$
part (4a)
We replace (1) and get
(3) $z(t) = v_0 t + \frac{1}{2}at^2$
(4)$\frac{d}{dt}z(t) = v_0 + at$
plug in (3),(4) in the Lagrangian
$L = \frac{1}{2}m\left(v_0 + at\right)^2 - mg\left(v_0 t + \frac{1}{2}at^2\right)$
$S = \int_{t_0}^{t_1} Ldt = m\int_{t_0}^{t_1}\left[\frac{1}{2}\left(v_0 + at\right)^2 - g\left(v_0 t + \frac{1}{2}at^2\right)\right]dt$
$S = m\int_{t_0}^{t_1}\left[\frac{1}{2}\left(v_0^2 + a^2t^2 + 2v_0 at\right) - gv_0 t - \frac{1}{2}gat^2\right]dt$

$S = \frac{1}{2}m\int_{t_0}^{t_1}\left[v_0^2 + 2v_0(a-g)t + a(a-g)t^2\right]dt$
$S = \frac{1}{2}m\left[v_0^2(t_1 - t_0) + v_0(a-g)(t_1-t_0)^2 + \frac{1}{3}a(a-g)(t_1^3 - t_0)^3\right]$
$t_0 = 0$ is replaced in the expression
we can replace $v_0$by noting that $v_0 = \frac{z(t)}{t} - \frac{1}{2}at$
now recall (2) and replace $z = z_1 \Rightarrow v_0 = \frac{z_1}{t_1} - \frac{1}{2}at_1$
for mere convenience t_1, z_1 its replaced with t, z, and
$S = \frac{1}{2}m\left[(\frac{z}{t} - \frac{1}{2}at)^2 t + (\frac{z}{t} - \frac{1}{2}at)(a-g)t^2 + \frac{1}{3}a(a-g)t^3\right]$
$S = \frac{1}{2}m\left[(\frac{z}{t} - \frac{1}{2}at)t\left[(\frac{z}{t} - \frac{1}{2}at) + (a-g)t\right] + \frac{1}{3}a(a-g)t^3\right]$
$S = \frac{1}{2}m\left[(\frac{z}{t} - \frac{1}{2}at)t\left[(\frac{z}{t} - \frac{1}{2}at + (a-g)t\right] + \frac{1}{3}a(a-g)t^3\right]$
$S = \frac{1}{2}m\left\{(z - \frac{1}{2}at^2)\left[(\frac{z}{t} - \frac{1}{2}at + (a-g)t\right] + \frac{1}{3}a(a-g)t^3\right\}$
$S = \frac{1}{2}m\left[\frac{z^2}{t} - \frac{1}{2}azt + (a-g)zt - \frac{1}{2}azt + \frac{1}{4}a^2t^3 - \frac{1}{2}a(a-g)t^3 + \frac{1}{3}a(a-g)t^3\right]$
$S = \frac{1}{2}m\left[\frac{z^2}{t} - gzt + \frac{1}{4}a^2t^3 - \frac{1}{6}a(a-g)t^3\right]$
$S = \frac{1}{2}m\left[\frac{z^2}{t} - gzt + \frac{1}{4}a^2t^3 - \frac{1}{6}a^2t^3 - \frac{1}{6}agt^3\right]$
$S = \frac{1}{2}m\left\{\frac{z^2}{t} - gzt - (\frac{1}{12}a^2 + \frac{1}{6}ag)t^3\right\}$
part (4b)
The way is to minimize S is:
$\frac{dS}{da} = 0 = \frac{a}{6} - \frac{g}{6} \Rightarrow a = -g$
==============PROBLEM 5===============
We must Minimize $I = \int 2\pi y ds$
$ds = \sqrt{1 + x'^2}dy$
$I = 2\pi\int y\sqrt{1+x'^2}dy$
$F = y\sqrt{1+x'^2}$
the Euler eq is
$\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0$
since $\frac{\partial F}{\partial x} = 0 \Longrightarrow \frac{\partial F}{\partial x'} = const$
$\frac{\partial F}{\partial x'} = \frac{yx'}{\sqrt{1+x'^2}} = c$
$x' = \frac{dx}{dy} = \frac{c}{\sqrt{y^2 - c^2}}$
$x = c \times cosh^{-1}(\frac{y}{c}) + b$
$y = c \times cosh(\frac{x-b}{c})$

==============PROBLEM 6===============



$ds = rd\theta \Longrightarrow ds - rd\theta = 0$
$f = s - r\theta = 0$
the kinetic energy of the disk $T = \frac{1}{2}(m\dot{s}^2 + I\dot{\theta}^2)$
the potential energy $U = -mgz = -mgs\sin\alpha$
$L = T + U$
$I = \int(L + \lambda f)dt$
1 equation of constraint gives one λ
$\frac{d}{dt}(\frac{\partial L}{\partial \dot{s}}) - \frac{\partial L}{\partial s} = \lambda\frac{\partial f}{\partial s}$
$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) - \frac{\partial L}{\partial \theta} = \lambda\frac{\partial f}{\partial \theta}$
$\frac{d}{dt}(\frac{\partial L}{\partial \dot{s}}) = m\ddot{s}; \frac{\partial L}{\partial s} = mg\sin\alpha$
$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) = I\ddot{\theta}; \frac{\partial L}{\partial \theta} = 0$
$\frac{\partial f}{\partial s} = 1$
$\frac{\partial f}{\partial \theta} = -r$
$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) - \frac{\partial L}{\partial \theta} = -r\lambda \Longrightarrow I\ddot{\theta} = -\lambda r \Longrightarrow \lambda = -\frac{I\ddot{\theta}}{r}$
by plugging in $I = \frac{1}{2}mr^2$ and $r\ddot{\theta} = \ddot{s}$ we get
$\lambda = -\frac{1}{2}m\ddot{s}$
$\frac{d}{dt}(\frac{\partial L}{\partial \dot{s}}) - \frac{\partial L}{\partial s} = \lambda \Longrightarrow m\ddot{s} - mg\sin\alpha = -\frac{1}{2}m\ddot{s}$
$\frac{3}{2}m\ddot{s} = mg\sin\alpha \Longrightarrow \ddot{s} = \frac{2}{3}g\sin\alpha$
the force of constraint is $\lambda = -\frac{1}{2}m\ddot{s} = -\frac{1}{3}mg\sin\alpha$