1. Prove that the magnitude R of the position vector for the center of mass from an arbitrary origin is given by the equation

$$M^2 R^2 = M \sum_i m_i r_i^2 - 1/2 \sum_{ij} m_i m_j r_{ij}^2$$

1) $M\overrightarrow{R} = \sum_i m_i \overrightarrow{r_i}$
2) $M^2R^2 = (\sum_i m_i \overrightarrow{r_i})(\sum_i m_i \overrightarrow{r_i}) = \sum_{ij} m_i m_j \overrightarrow{r_i} \cdot \overrightarrow{r_j}$
3) $\|\overrightarrow{r_i} - \overrightarrow{r_j}\|^2 = r_{ij}^2 = r_i^2 + r_j^2 - 2\overrightarrow{r_i} \cdot \overrightarrow{r_j}$
4) $\frac{1}{2}\left(r_i^2 + r_j^2 - r_{ij}^2\right) = \overrightarrow{r_i} \cdot \overrightarrow{r_j}$
now let's plug 4) into 2) to substitute for $\overrightarrow{r_i} \cdot \overrightarrow{r_j}$
$M^2R^2 = \frac{1}{2}\sum_{ij} m_i m_j \left(r_i^2 + r_j^2 - r_{ij}^2\right)$
$M^2R^2 = \frac{1}{2}\sum_{ij} m_i m_j r_i^2 + \frac{1}{2}\sum_{ij} m_i m_j r_j^2 - \frac{1}{2}\sum_{ij} m_i m_j r_{ij}^2$
$M^2R^2 = \frac{1}{2}\sum_j m_j \sum_i m_i r_i^2 + \frac{1}{2}\sum_i m_i \sum_i m_j r_j^2 - \frac{1}{2}\sum_{ij} m_i m_j r_{ij}^2$
$M^2R^2 = \frac{1}{2}M\sum m_i r_i^2 + \frac{1}{2}M\sum_i m_j r_j^2 - \frac{1}{2}\sum_{ij} m_i m_j r_{ij}^2$
$M^2R^2 = M\sum_i m_i r_i^2 - \frac{1}{2}\sum_{ij} m_i m_j r_{ij}^2$
So $a = 1$, $b = -1/2$, and $r_{ij}^2 = \|\overrightarrow{r_i} - \overrightarrow{r_j}\|^2$
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2. Two wheels of radius a are mounted on the ends of a common axle of length b such that the wheels rotate independently. The whole combination rolls without slipping on a plane. Show that there are two nonholonomic equations of constraint,

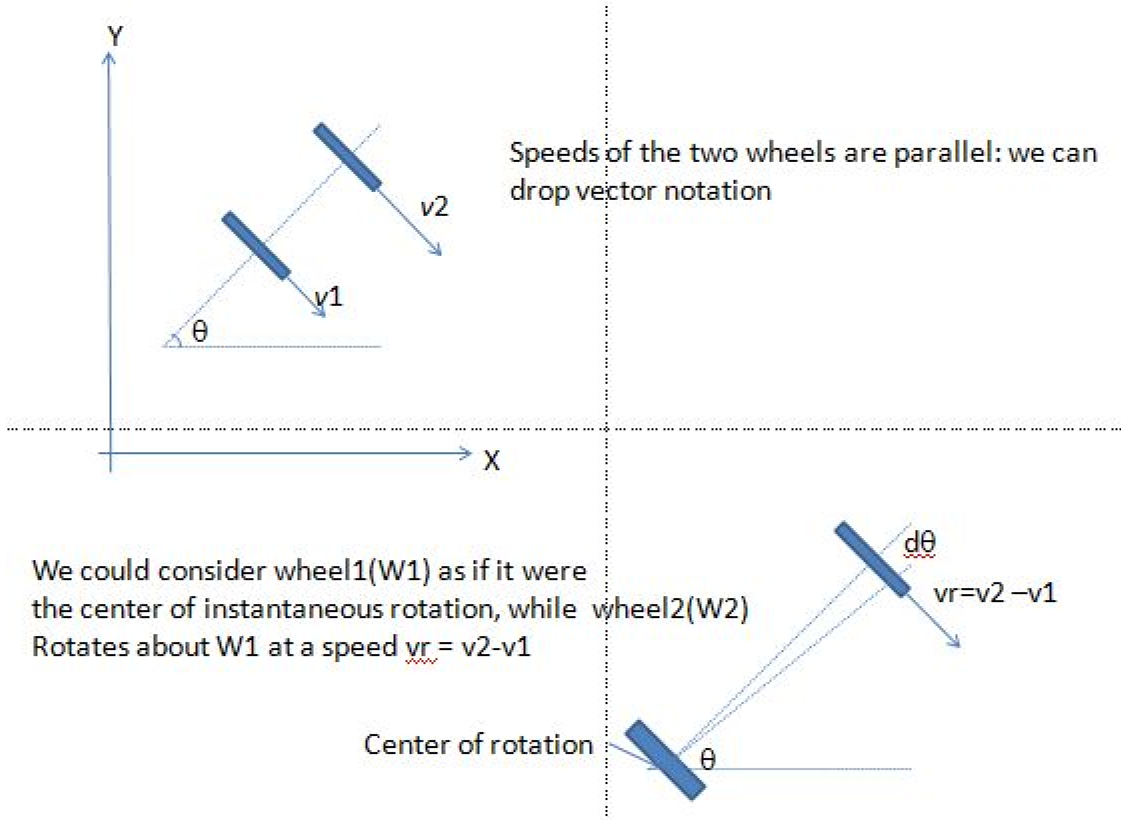
$$\cos\theta \; dx + \sin\theta \; dy = 0$$

$$\sin\theta \; dx - \cos\theta \; dy = a/2\,(d\phi + d\phi')$$

(where θ, ϕ, and ϕ' are similar in meaning as for the case of a simple disk rolling on a plane and x and y are the coordinates of a point on the axle midway between the two wheels) and one holonomic equation of constraint

$$\theta = C - a/b\,(\phi - \phi')$$

where C is a constant.



We can use the following 5 coordinates to describe the system:
center of mass (x,y), orientation angle (θ), angles ϕ_1,ϕ_2
We have the 4 constraints:
$\overrightarrow{v_1} = -a\dot{\phi}_1\hat{r_\perp}$ (1) and $\overrightarrow{v_2} = -a\dot{\phi}_2\hat{r_\perp}$ (2),
where $\hat{r} = (\cos\Theta, \sin\Theta)$, $\hat{r_\perp} = (-\sin\Theta, \cos\Theta)$

Now $\overrightarrow{v_{CM}} = \frac{\overrightarrow{v_1}+\overrightarrow{v_2}}{2} \propto \hat{r_\perp}$ and thus $\overrightarrow{v_{CM}} \cdot \hat{r} = 0$ (3)
Also $\overrightarrow{v_2} - \overrightarrow{v_1} \propto \hat{r_\perp}$ and thus $\overrightarrow{v_2} - \overrightarrow{v_1} = b\dot{\Theta}\hat{r_\perp}$ (5).

In addition using (1) and (2) we find$\overrightarrow{v_{CM}} \cdot \hat{r_\perp} = -\frac{a\dot{\phi}_1+a\dot{\phi}_2}{2}$ (6).
Now note that $\overrightarrow{v_{CM}} = (\dot{x}, \dot{y})$.
Therefore (3) yields $\dot{x}\cos\Theta + \dot{y}\sin\Theta = 0$ (3b),
while (6) gives $-\dot{x}\sin\Theta + \dot{y}\cos\Theta = -\frac{a}{2}(\dot{\phi}_1 + \dot{\phi}_2)$ (6b).

From (5) we get (using (1) and (2)) $-a(\dot{\phi}_2 - \dot{\phi}_1) = b\dot{\Theta}$ (5b).

(3b) and (6b) can be written as
$dx\cos\Theta + dy\sin\Theta = 0$ and $dx\sin\Theta - dy\cos\Theta = \frac{a}{2}(d\phi_1 + d\phi_2)$.
(5b) can be integrated to yield $a(\phi_1 - \phi_2) = b\Theta - C$.
qed.
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4. If L is a Lagrangian for a system of n degrees of freedom satisfying Lagrange's equations, show by direct substitution that

$$L' = L + \frac{dF(q_1,\ldots,q_n,t)}{dt}$$

also satisfies Lagrange's equations where F is any arbitrary, but differentiable, functions of its arguments.

3.

1) $\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) - \frac{\partial L'}{\partial q_i} = 0$

$\frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i}\left(L + \frac{dF}{dt}\right) = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial \dot{F}}{\partial \dot{q}_i}$

Note: $\dot{F} = \frac{dF}{dt} = \sum_i \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial t}$. Thus $\frac{\partial \dot{F}}{\partial \dot{q}_i} = \frac{\partial F}{\partial q_i}$ as well as $\frac{\partial}{\partial q_j}\frac{dF}{dt} = \sum_i \frac{\partial^2 F}{\partial q_i \partial q_j}\dot{q}_i + \frac{\partial^2 F}{\partial t \partial q_j}$.

We also note that $\frac{d}{dt}\frac{\partial F}{\partial q_j} = \sum_i \frac{\partial^2 F}{\partial q_j \partial q_i}\dot{q}_i + \frac{\partial^2 F}{\partial q_j \partial t}$, so that really $\frac{\partial}{\partial q_j}\frac{dF}{dt} = \frac{d}{dt}\frac{\partial F}{\partial q_j}$.

The use of $\frac{\partial \dot{F}}{\partial \dot{q}_i} = \frac{\partial F}{\partial q_i}$ leads to

$\frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial F}{\partial q_i}$

2) $\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} + \frac{\partial F}{\partial q_i}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{d}{dt}\left(\frac{\partial F}{\partial q_i}\right)$

3) $\frac{\partial L'}{\partial q_i} = \frac{\partial}{\partial q_i}\left(L + \frac{dF}{dt}\right) = \frac{\partial L}{\partial q_i} + \frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right)$

we plug in 2) , 3) into 1)

4) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{d}{dt}\left(\frac{\partial F}{\partial q_i}\right) - \frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right)$

substituting $\frac{d}{dt}\left(\frac{\partial F}{\partial q_i}\right) = \frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right)$ into 4)

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right) - \frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}$

which proves that

$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) - \frac{\partial L'}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}$

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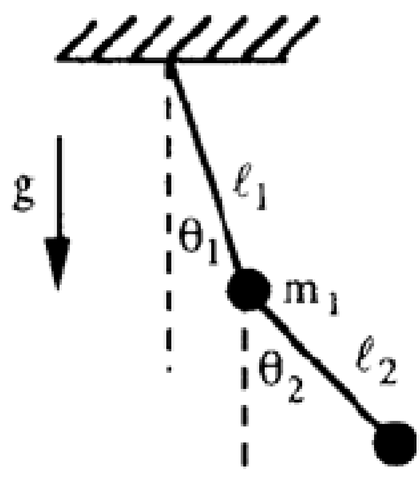
4. using Lagrangian we can find the equations of motion for the double pendulum



a)

let's assume $q_1 = \theta_1$and $q_2 = \theta_2$

$\overrightarrow{v_1} = l_1\dot{\theta}_1\hat{\theta}_1$

$v_1^2 = l_1^2\dot{\theta}_1^2$

$\overrightarrow{v_2} = l_1\dot{\theta}_1\hat{\theta}_1 + l_2\dot{\theta}_2\hat{\theta}_2$
$v_2^2 = l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_1\overrightarrow{\dot{\theta}_1} \cdot \overrightarrow{\dot{\theta}_2}$
$v_2^2 = l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_1\dot{\theta}_1\dot{\theta}_2cos(\theta_1 - \theta_2)$
$T = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1l_1^2\dot{\theta_1}^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta_1}^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2cos(\theta_1 - \theta_2)\right]$
$U = -(gm_1l_1cos\theta_1 + gm_2l_1cos\theta_1 + gm_2l_2cos\theta_2)$
$L = T - U = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1l_1^2\dot{\theta_1}^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta_1}^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2cos(\theta_1 - \theta_2)\right] + (m_1 + m_2)gl_1cos(\theta_1) + m_2gl_2cos(\theta_2)$
b)
$\frac{\partial L}{\partial\dot{\theta_1}} = m_1l_1^2\dot{\theta}_1 + m_2l_1^2\dot{\theta}_1 + m_2l_1l_2\dot{\theta}_2cos(\theta_1 - \theta_2)$
$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta_1}}\right) = l_1^2\ddot{\theta}_1(m_1 + m_2) + m_2l_1l_2\ddot{\theta}_2cos(\theta_1 - \theta_2) - m_2l_1l_2\dot{\theta}_2(\dot{\theta}_1 - \dot{\theta}_2)sin(\theta_1 - \theta_2)$
$\frac{\partial L}{\partial\theta_1} = -m_2l_1l_2\dot{\theta}_1\dot{\theta}_2sin(\theta_1 - \theta_2) - (m_1 + m_2)gl_1sin(\theta_1)$
$\frac{\partial L}{\partial\dot{\theta_2}} = m_2l_2^2\dot{\theta}_2 + m_2l_1l_2\dot{\theta}_1cos(\theta_1 - \theta_2)$
$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta_2}}\right) = m_2l_2^2\ddot{\theta}_2 + m_2l_1l_2\ddot{\theta}_1cos(\theta_1 - \theta_2) - m_2l_1l_2\dot{\theta}_1(\dot{\theta}_1 - \dot{\theta}_2)sin(\theta_1 - \theta_2)$
$\frac{\partial L}{\partial\theta_2} = m_2l_1l_2\dot{\theta}_1\dot{\theta}_2sin(\theta_1 - \theta_2) - gm_2l_2sin\theta_2$
First eq of motion is
$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta_1}}\right) - \frac{\partial L}{\partial\theta_1} = 0$
$l_1^2\ddot{\theta}_1(m_1 + m_2) + m_2l_1l_2\ddot{\theta}_2cos(\theta_1 - \theta_2) + m_2l_1l_2\dot{\theta}_2^2sin(\theta_1 - \theta_2) - m_2l_1l_2\dot{\theta}_2\dot{\theta}_1sin(\theta_1 - \theta_2) + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2sin(\theta_1 - \theta_2) + (m_1 + m_2)gl_1sin(\theta_1)$
1) $l_1\ddot{\theta}_1(m_1 + m_2) + m_2l_1l_2\ddot{\theta}_2cos(\theta_1 - \theta_2) + m_2l_1l_2\dot{\theta}_2^2sin(\theta_1 - \theta_2) + (m_1 + m_2)gl_1sin(\theta_1) = 0$
the second eq of motion is
$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta_2}}\right) - \frac{\partial L}{\partial\theta_2} = 0$
2) $m_2l_2^2\ddot{\theta}_2 + m_2l_1l_2\ddot{\theta}_1cos(\theta_1 - \theta_2) - ml_1l_2\dot{\theta}_1^2sin(\theta_1 - \theta_2) + gm_2l_2sin\theta_2 = 0$
these two equations are coupled equations
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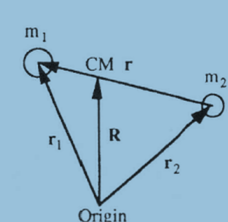
7. The Lagrangian for two particles of masses m_1 and m_2 and coordinates $\mathbf{r_1}$ and $\mathbf{r_2}$ that interact via a potential $V(\mathbf{r_1} - \mathbf{r_2})$ is

$L = ½\ m_1\ |d\,\mathbf{r_1}/dt|^2 + ½\ m_2\ |d\,\mathbf{r_2}/dt|^2 - V(\mathbf{r_1} - \mathbf{r_2})$

a) Rewrite the Lagrangian in terms of the center of mass coordinates R and relative coordinates $\mathbf{r} = \mathbf{r_1} - \mathbf{r_2}$.

b) Use Lagrange's equations to show that the center of mass and relative motions separate, the center of mass moving with constant velocity and the relative motion being that of a particle of reduced mass $\eta = (m_1\ m_2)/(m_1 + m_2)$ in a potential $V(\mathbf{r})$.



It is assumed that we are working with vectors and can avoid using arrows notation.
$r = r_1 - r_2$
$MR = m_1r_1 + m_2r_2$
$MR = mr_1 + m(r_1 - r)$
$MR = r_1(m_1 + m_2) - m_2r$
1) $r1 = R + \frac{m_2}{M}r$

$MR = m_1(r_2 + r) + m_2 r_2$
$MR = r_2(m_1 + m_2) + m_1 r$
2) $r_2 = R - \frac{m_1}{M} r$
$T = \frac{1}{2}(m_1 \dot{r_1}^2 + m_2 \dot{r_2}^2)$
3)$\dot{r_1}^2 = (\dot{R} + \frac{m_2}{M}\dot{r})^2$
4)$\dot{r_2}^2 = (\dot{R} - \frac{m_1}{M}\dot{r})^2$
replace 3) and 4) in the kinetic energy
$T = \frac{1}{2}\left[m_1\left(\dot{R} + \frac{m_2}{M}\dot{r}\right)^2 + m_2\left(\dot{R} - \frac{m_1}{M}\dot{r}\right)^2\right]$
$m_1\left(\dot{R} + \frac{m_2}{M}\dot{r}\right)^2 = m_1\dot{R}^2 + \dot{r}^2\frac{m_1 m_2^2}{M^2} + \frac{2m_1 m_2}{M}(\dot{R}\cdot\dot{r})$
$m_2(R - \frac{m_1}{M}r)^2 = m_2\dot{R}^2 + \dot{r}^2\frac{m_2 m_1^2}{M^2} - \frac{2m_1 m_2}{M}(\dot{R}\cdot\dot{r})$
$T = \frac{1}{2}\left[\dot{R}^2(m_1 + m_2) + \dot{r}^2\left(\frac{m_1 m_2}{M^2}\right)(m_1 + m_2)\right]$
$\mu = \left(\frac{m_1 m_2}{m_1 + m_2}\right)$ is the reduced mass
$T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu\dot{r}^2$
finally the Lagrangian is
L = T - U
$L = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu\dot{r}^2 - U(r)$
$\partial L_{\dot{R}} = M\dot{R}$
$\partial L_R = 0$
$\frac{d}{dt}(\partial L_{\dot{R}}) = M\ddot{R}$
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{R}}\right) - \frac{\partial L}{\partial R} = 0$
$M\ddot{R} = 0$ which means the velocity of the center of mass is a constant of the motion
$\partial L_{\dot{r}} = \mu\dot{r}$
$\frac{d}{dt}(\partial L_{\dot{r}}) = \mu\ddot{r}$
$\partial L_r = -\partial U(r)$
$\mu\ddot{r} = -\partial U(r)$
Like a particle of mass μmoving in a potential U(r)

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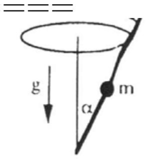

6. A bead of mass m slides along a straight wire which makes an angle and rotates with constant angular velocity . Find the Lagrange's equations of motion.

a)
Let's assume $q_1 = R$ and $q_2 = \theta$, where R is the displacement of the bead measured along the wire.
$\dot{\theta} = \omega$
$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{R}^2 + \omega^2 R^2 \sin^2\alpha)$
$U = mgR\cos\alpha$
$L = T - U = \frac{1}{2}m(\dot{R}^2 + \omega^2 R^2 \sin^2\alpha) - mgR\cos\alpha$
b)

$\partial L/\partial \dot{R} = m\dot{R}$
$\frac{d}{dt}(\partial L/\partial \dot{R}) = m\ddot{R}$
$\partial L/\partial R = m\omega^2 R \sin^2\alpha - mg\cos\alpha$
the eq of motion is:
$\frac{d}{dt}(\partial L/\partial \dot{R}) - \partial L/\partial R = 0$
$m\ddot{R} - m\omega^2 R \sin^2\alpha + mg\cos\alpha = 0$
==
ALTERNATIVE: polar coords:
use $z = R\cos\alpha$
$L = \frac{1}{2}m(\frac{\dot{z}^2}{\cos^2\alpha} + \omega^2 z^2 \tan^2\alpha) - mgRz$