Numerical Relativity - PHY 6938

Solutions to HW 6

1. This problem is actually quite lengthy. Here we give only the outline of a straightforward brute force attack.
Start with

$$R_{ab} = \partial_c \Gamma^c_{ab} - \partial_a \Gamma^c_{cb} + \Gamma^c_{ab}\Gamma^d_{cd} - \Gamma^c_{db}\Gamma^d_{ca}$$

rewrite in terms of Γ_{abc}:

$$R_{ab} = \partial_c(g^{ce}\Gamma_{eab}) - \partial_a(g^{ce}\Gamma_{ecb}) + g^{ce}\Gamma_{eab}g^{df}\Gamma_{fcd} - g^{ce}\Gamma_{edb}g^{df}\Gamma_{fca}$$

If we insert the definition of Γ_{abc}, we get lots of terms with first and second derivatives of the metric g_{ab}, but we also have derivatives of the inverse metric of the form $\partial_a g^{ce}$ from the first 2 terms in R_{ab}. Convert these terms into derivatives of the metric using

$$\partial_a g^{bc} = -g^{bd}g^{ce}\partial_a g_{de}.$$

After this we have only the inverse metric g^{ce} and first and second derivatives of the metric g_{de} in R_{ab}.
Now use the same approach on the equation

$$R'_{ab} = -\frac{1}{2}g^{cd}\partial_c\partial_d g_{ab} + \nabla_{(a}\Gamma_{b)} + g^{cd}g^{ef}\left(\partial_e g_{ca}\partial_f g_{db} - \Gamma_{ace}\Gamma_{bdf}\right)$$

as given in the problem. Again we obtain an expression in terms of the inverse metric g^{ce} and first and second derivatives of the metric g_{de} in R'_{ab}. If we subtract R_{ab} from R'_{ab} and rename some dummy indices we obtain zero. This proves that the expression R'_{ab} is indeed the Ricci tensor.

2. $\Box x^\alpha = g^{\mu\nu}\nabla_\mu\nabla_\nu x^\alpha$.
a) $\Box x^\alpha = \frac{1}{\sqrt{|g|}}\partial_\mu \mathfrak{g}^{\mu\alpha}$
b) $\Box x^\alpha = g^{\mu\nu}(\partial_\mu\nabla_\nu x^\alpha - \Gamma^\lambda_{\mu\nu}\nabla_\lambda x^\alpha) = g^{\mu\nu}(\partial_\mu\delta^\alpha_\nu - \Gamma^\lambda_{\mu\nu}\delta^\alpha_\lambda) = -g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = -\Gamma^\alpha$
c) Note that $\nabla^\mu n_\mu = \nabla^\mu(-\alpha\nabla_\mu t) = -\alpha\nabla^\mu\nabla_\mu t - (\nabla^\mu\alpha)\nabla_\mu t = n_\nu\Box x^\nu + (n^\mu\nabla_\mu\alpha)/\alpha$
We then find

$$(\pounds_t - \pounds_\beta)\alpha = -\alpha^2(n_\mu\Box x^\mu + K),$$

where $n_\mu\Box x^\mu = -\alpha\Box x^t$
d) First note that for any scalar function f: $\gamma^{\mu\nu}\nabla_\mu\nabla_\nu f = \gamma^{\mu\nu}g^{\mu'}_\mu\nabla_{\mu'}(g^{\nu'}_\nu\nabla_{\nu'}f) = \gamma^{\mu\nu}\gamma^{\mu'}_\mu\nabla_{\mu'}([\gamma^{\nu'}_\nu - n_\nu n^{\nu'}]\nabla_{\nu'}f) = \gamma^{\mu\nu}\gamma^{\mu'}_\mu[\nabla_{\mu'}(\gamma^{\nu'}_\nu\nabla_{\nu'}f) - \nabla_{\mu'}(n_\nu n^{\nu'}\nabla_{\nu'}f)] = \gamma^{\mu\nu}[D_\mu D_\nu f - \gamma^{\mu'}_\mu\nabla_{\mu'}(n_\nu n^{\nu'})(\nabla_{\nu'}f) - 0] = \gamma^{\mu\nu}[D_\mu D_\nu f - \gamma^{\mu'}_\mu[(\nabla_{\mu'}n_\nu)n^{\nu'} + 0](\nabla_{\nu'}f)] = \gamma^{\mu\nu}D_\mu D_\nu f - \gamma^{\mu\nu}\gamma^{\mu'}_\mu(\nabla_{\mu'}n_\nu)n^{\nu'}(\nabla_{\nu'}f) = \gamma^{\mu\nu}D_\mu D_\nu f + Kn^{\nu'}\nabla_{\nu'}f = \gamma^{kl}D_k D_l f + Kn^\nu\nabla_\nu f$.
For $f = x^i$ we thus get: $\gamma^{\mu\nu}\nabla_\mu\nabla_\nu x^i = \gamma^{kl}D_l\partial_k x^i + Kn^\nu\nabla_\nu x^i = \gamma^{kl}(\partial_l\partial_k x^i - {}^{(3)}\Gamma^j_{kl}\partial_j x^i) + Kn^\nu\nabla_\nu x^i = \gamma^{kl}(\partial_l\delta^i_k - {}^{(3)}\Gamma^j_{kl}\delta^i_j) + Kn^\nu\nabla_\nu x^i = -\gamma^{kl}\,{}^{(3)}\Gamma^i_{kl} + Kn^\nu\nabla_\nu x^i = -{}^{(3)}\Gamma^i + Kn^\nu\nabla_\nu x^i$.
Also $n^\mu n^\nu\nabla_\mu\nabla_\nu x^i = n^\mu\nabla_\mu(n^\nu\nabla_\nu x^i) - (n^\mu\nabla_\mu n^\nu)\nabla_\nu x^i$, where $n^\mu\nabla_\mu n^\nu = D^\nu \ln\alpha$.

Now note that $n^\nu \nabla_\nu x^i$ is a scalar with the value $n^\nu \nabla_\nu x^i = n^\nu \delta^i_\nu = n^i = -\beta^i/\alpha$. Thus $n^\mu \nabla_\mu (n^\nu \nabla_\nu x^i) = n^\mu \partial_\mu (n^i) = n^0 \partial_t n^i + n^k \partial_k n^i = -\frac{1}{\alpha^2}\partial_t \beta^i + \frac{\beta^i}{\alpha^3}\partial_t \alpha - \frac{\beta^k}{\alpha}(-\frac{1}{\alpha}\partial_k \beta^i + \frac{\beta^i}{\alpha^2}\partial_k \alpha) = \frac{1}{\alpha^2}(-\partial_t \beta^i + \beta^i \partial_t \ln\alpha + \beta^k \partial_k \beta^i - \beta^k \beta^i \partial_k \ln\alpha) = \frac{1}{\alpha^2}(-\partial_t \beta^i + \beta^k \partial_k \beta^i) + \frac{\beta^i}{\alpha^2}(\partial_t \ln\alpha - \beta^k \partial_k \ln\alpha) = \frac{\beta^i}{\alpha^2} d_0 \ln\alpha - \frac{1}{\alpha^3} d_0 \beta^i$ where $d_0 := \partial_t - \beta^k \partial_k$.
Putting all this together $n^\mu n^\nu \nabla_\mu \nabla_\nu x^i = \frac{\beta^i}{\alpha^3}(d_0 \alpha) - \frac{1}{\alpha^2} d_0 \beta^i - D^i \ln\alpha$, so finally $\Box x^i = -{}^{(3)}\Gamma^i + K n^\nu \nabla_\nu x^i - \frac{\beta^i}{\alpha^3} d_0 \alpha + \frac{1}{\alpha^2} d_0 \beta^i + D^i \ln\alpha$, which we rewrite as

$$d_0 \beta^i = \alpha^2 (\Box x^i + {}^{(3)}\Gamma^i + K\beta^i/\alpha) + \frac{1}{\alpha}\beta^i d_0 \alpha - \alpha D^i \alpha$$

Note that $\gamma^i_\mu \Box x^\mu = (g^i_\mu + n^i n_\mu)\Box x^\mu = \Box x^i + n^i n_\mu \Box x^\mu$. Thus using $n_\mu \Box x^\mu = -\frac{1}{\alpha^2}(\pounds_t - \pounds_\beta)\alpha - K$ form 2.c) we find

$$\gamma^i_\mu \Box x^\mu = \Box x^i + (\beta^i/\alpha)(\frac{1}{\alpha^2} d_0 \alpha + K) = \Box x^i + K\beta^i/\alpha + \frac{\beta^i}{\alpha^3} d_0 \alpha$$

Thus $d_0 \beta^i = \alpha^2 (\gamma^i_\mu \Box x^\mu - \frac{\beta^i}{\alpha^3} d_0 \alpha + {}^{(3)}\Gamma^i) + \frac{1}{\alpha}\beta^i d_0 \alpha - \alpha D^i \alpha = \alpha^2 (\gamma^i_\mu \Box x^\mu + {}^{(3)}\Gamma^i) - \alpha D^i \alpha$.
We now rewrite ${}^{(3)}\Gamma^i = \gamma^{kl}\ {}^{(3)}\Gamma^i_{kl} = \gamma^{kl}\gamma^{ij}\ {}^{(3)}\Gamma_{jkl} = \gamma^{kl}\gamma^{ij}\Gamma_{jkl} = \gamma^{i\rho}\gamma^{\mu\nu}\Gamma_{\rho\mu\nu}$ which holds because ${}^{(3)}\Gamma_{jkl} = \Gamma_{jkl}$, which in turn is true because $\gamma_{kl} = g_{kl}$. So finally we get:

$$(\partial_t - \beta^k \partial_k)\beta^i = \alpha^2 (\gamma^i_\nu \Box x^\nu + \gamma^{ij}\gamma^{kl}\Gamma_{jkl}) - \alpha \gamma^{ij}\partial_j \alpha$$

3. Generalized Harmonic formulation:
Choose coordinates such that

$$g_{\alpha\beta}\Box x^\beta = H_\alpha,$$

where $H_\alpha = H_\alpha(t, x^i, g_{\mu\nu})$ are functions that can be freely chosen.
a) Using our results from 1. & 2. we find:

$$(\partial_t - \beta^k \partial_k)\alpha = -\alpha(H_t - \beta^i H_i + \alpha K)$$

$$(\partial_t - \beta^k \partial_k)\beta^i = \alpha\gamma^{ij}[\alpha(H_j + \gamma^{kl}\Gamma_{jkl}) - \partial_j \alpha]$$

b) The Einstein equations are

$$-\frac{1}{2} g^{cd}\partial_c \partial_d g_{ab} - \nabla_{(a} H_{b)} + g^{cd} g^{ef}\left(\partial_e g_{ca}\partial_f g_{db} - \Gamma_{ace}\Gamma_{bdf}\right) = 8\pi\left(T_{ab} - \frac{1}{2} g_{ab} T\right)$$

where we have the constraint

$$\Gamma_\alpha = -H_\alpha.$$

c) We choose coodinates by specifying H_α. The resulting system of equations is manifestly symmetric hyperbolic because it has the same principal terms (highest derivative terms) as a wave equation for each metric component g_{ab}. This is important, because it implies that the system is well-posed.