Numerical Relativity - PHY 6938

Solutions to HW 5

1. Recall that two covariant derivative operators differ by a tensor C^l_{ik}, e.g.

$$D_i w_k = \bar{D}_i w_k - C^l_{ik} w_l.$$

Read Wald's book if you do not believe this.
Let's start with

$$R_{ijk}{}^l w_l = D_i D_j w_k - D_j D_i w_k$$

and note $D_i D_j w_k = \bar{D}_i(\bar{D}_j w_k - C^m_{jk} w_m) - C^l_{ij}(\bar{D}_l w_k - C^m_{lk} w_m) - C^l_{ik}(\bar{D}_j w_l - C^m_{jl} w_m)$.
Thus $R_{ijk}{}^l w_l = D_i D_j w_k - D_j D_i w_k = \bar{R}_{ijk}{}^l w_l + (2\bar{D}_{[j} C^l_{i]k} + 2C^m_{k[i} C^l_{j]m}) w_l$.
If $D_i \gamma_{kl} = 0$ and $\bar{D}_i \bar{\gamma}_{kl} = 0$, we have

$$C^k_{ij} = \frac{1}{2}\gamma^{kl}(\bar{D}_j \gamma_{li} + \bar{D}_i \gamma_{lj} - \bar{D}_l \gamma_{ij}).$$

We use this for the case of $\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$, $\gamma^{ij} = \psi^{-4}\bar{\gamma}^{ij}$, and find $\bar{D}_i \gamma_{jk} = 4\psi^3 \bar{\gamma}_{jk} \bar{D}_i \psi = 4\gamma_{jk} D_i \ln\psi$, which leads to

$$C^l_{ij} = (4\delta^l_{(i}\bar{D}_{j)} - 2\bar{\gamma}_{ij}\bar{\gamma}^{kl}\bar{D}_k)\ln\psi.$$

Then the Riemann tensor is $R_{ijk}{}^l = \bar{R}_{ijk}{}^l + 2\bar{D}_{[j} C^l_{i]k} + 2C^m_{k[i} C^l_{j]m} = \bar{R}_{ijk}{}^l + 4\delta^l_{[i}\bar{D}_{j]}\bar{D}_k \ln\psi - 4\bar{\gamma}^{lm}\bar{\gamma}_{k[i}\bar{D}_{j]}\bar{D}_m \ln\psi + 8(\bar{D}_{[i}\ln\psi)\delta^l_{j]}\bar{D}_k \ln\psi - 8(\bar{D}_{[i}\ln\psi)\bar{\gamma}_{j]k}\bar{\gamma}^{lm}\bar{D}_m \ln\psi - 8\bar{\gamma}_{k[i}\delta^l_{j]}\bar{\gamma}^{mn}(\bar{D}_m \ln\psi)\bar{D}_n \ln\psi$.
So the Ricci tensor becomes $R_{ik} = R_{ilk}{}^l = \bar{R}_{ik} - 2\bar{D}_i\bar{D}_k \ln\psi - 2\bar{\gamma}_{ik}\bar{\gamma}^{lm}\bar{D}_l\bar{D}_m \ln\psi + 4(\bar{D}_i \ln\psi)\bar{D}_k \ln\psi - 4\bar{\gamma}_{ik}\bar{\gamma}^{lm}(\bar{D}_l \ln\psi)\bar{D}_m \ln\psi$.
Form this we can compute the Ricci scalar and find

$$R = \psi^{-4}\bar{R} - 8\psi^{-5}\bar{D}_k\bar{D}^k\psi.$$

[Note: $\bar{D}_i\bar{D}_j \ln\psi = \bar{D}_i(\psi^{-1}\bar{D}_j\psi) = -\psi^{-2}(\bar{D}_i\psi)\bar{D}_j\psi + \psi^{-1}\bar{D}_i\bar{D}_j\psi$. So $\psi^{-1}\bar{D}_i\bar{D}_j\psi = \bar{D}_i\bar{D}_j \ln\psi + \psi^{-2}(\bar{D}_i\psi)\bar{D}_j\psi = \bar{D}_i\bar{D}_j \ln\psi + (\bar{D}_i \ln\psi)(\bar{D}_j \ln\psi)$.]

2. Let's start with $D_j(\psi^n S^{ij}) = n\psi^{n-1} S^{ij} D_j\psi + \psi^n D_j S^{ij}$ and replace D_j by $\bar{D}_j$ and C^l_{ij} as in 1.
Take into account that S^{ij} is symmetric. We then find

$$D_j(\psi^n S^{ij}) = \psi^n[\bar{D}_j S^{ij} + (10+n) S^{ij}\bar{D}_j \ln\psi - 2\bar{\gamma}_{jk} S^{jk}\bar{D}^i \ln\psi].$$

3. Since $(LW)^{ij} := D^i W^j + D^j W^i - (2/3)\gamma^{ij} D_k W^k$, we need $D^i W^j = \gamma^{im} D_m W^j = \gamma^{im}(\bar{D}_m W^j + C^j_{mk} W^k)$.
With the C^j_{mk} from 1. we find:

$$(LW)^{ij} = \psi^{-4}(\bar{L}W)^{ij}$$

4. We have $\gamma_{ij} = \psi^4\delta_{ij}$. Thus $\bar{D}_i = \partial_i$. Now from 3. we find $(L\beta)^{ij} = \psi^{-4}(\bar{L}\beta)^{ij} = \psi^{-4}(\partial^i\beta^j + \partial^j\beta^i - (2/3)\delta^{ij}\partial_k\beta^k)$.
Here we have $\beta^i = \epsilon^{ijk}\Omega^j x^k$ (where we sum over j and k). Thus $(L\beta)^{ij} = 0$.

5. If $\gamma_{ij} = \psi^4\delta_{ij}$ the ADM mass is given by

$$M_{ADM} = -\frac{1}{2\pi}\lim_{r\to\infty}\oint_S \partial_j\psi dS^j.$$

If we construct puncture initial data and solve the Hamiltonian constraint for u we obtain $\psi = 1 + \sum_A \frac{m_A}{2r_A} + u$ as conformal factor. This results in

$$M_{ADM} = \sum_A m_A - \frac{1}{2\pi}\lim_{r\to\infty}\oint_S \partial_j u dS^j = \sum_A m_A - \frac{1}{2\pi}\lim_{r\to\infty}\int_0^\pi\int_0^{2\pi}(\partial_r u)\sin\theta\; d\theta d\phi.$$

[Im letzten Integral fehlt ein Faktor r^2.]