Numerical Relativity - PHY 6938

Solutions to HW 4

1. Let's start with ${}^{(3)}R_{abc}{}^{d} w_d = D_a D_b w_c - D_b D_a w_c$ and note $D_a D_b w_c = \gamma_a^f \gamma_b^g \gamma_c^h \nabla_f (\gamma_g^d \gamma_h^e \nabla_d w_e)$. Now we use $\gamma_a^f \gamma_b^g \nabla_f \gamma_g^d = \gamma_a^f \gamma_b^g \nabla_f (\delta_g^d + n_g n^d) = \gamma_a^f \gamma_b^g (n^d \nabla_f n_g + n_g \nabla_f n^d) = \gamma_a^f \gamma_b^g (n^d \nabla_f n_g + 0) = K_{ab} n^d$. So $D_{[a} D_{b]} w_c = K_{c[a} n^e \gamma_{b]}^g \gamma_g^d \nabla_d w_e + \gamma_{[a}^f \gamma_{b]}^g \gamma_c^h \gamma_g^d \gamma_h^e \nabla_f \nabla_d w_e = K_{c[a} \gamma_{b]}^d n^e \nabla_d w_e + \gamma_{[a}^f \gamma_{b]}^d \gamma_c^h \gamma_h^e \nabla_f \nabla_d w_e$. Now use $n^e w_e = 0$ to obtain $\gamma_b^d n^e \nabla_d w_e = -\gamma_b^d w_e \nabla_d n^e = K_b{}^e w_e$. Thus $2 D_{[a} D_{b]} w_c = 2 K_{c[a} K_{b]}{}^e w_e + 2 \gamma_a^f \gamma_b^d \gamma_c^h \gamma_h^e \nabla_{[f} \nabla_{d]} w_e = (K_{ca} K_{be} - K_{cb} K_{ae} + \gamma_a^f \gamma_b^d \gamma_c^h \gamma_{he} 2 \nabla_{[f} \nabla_{d]}) w^e = (K_{ca} K_{be} - K_{cb} K_{ae}) w^e + \gamma_a^f \gamma_b^d \gamma_{ce} 2 \nabla_{[f} \nabla_{d]} w^e$, which gives ${}^{(3)}R_{abcd} w^d = 2 D_{[a} D_{b]} w_c = (K_{ac} K_{be} - K_{bc} K_{ae} + \gamma_a^f \gamma_b^d \gamma_{ce'} R_{fd}{}^{e'}{}_e) w^e = (K_{ac} K_{bh} - K_{bc} K_{ah} + \gamma_a^e \gamma_b^f \gamma_c^g R_{efgh}) w^h$ and thus using $w^h = \gamma_i^h w^i$ we finally find the Gauß-Codazzi relation

$$ {}^{(3)}R_{abcd} = K_{ac} K_{bd} - K_{bc} K_{ad} + \gamma_a^e \gamma_b^f \gamma_c^g \gamma_d^h R_{efgh}. $$

Next we start with $D_a K_{bc} = \gamma_a^d \gamma_b^e \gamma_c^f \nabla_d (-\gamma_e^g \nabla_g n_f) = -\gamma_a^d \gamma_b^e \gamma_c^f \gamma_e^g \nabla_d \nabla_g n_f - \gamma_a^d \gamma_b^e \gamma_c^f (\nabla_d \gamma_e^g)(\nabla_g n_f) = -\gamma_a^d \gamma_b^e \gamma_c^f \gamma_e^g \nabla_d \nabla_g n_f - \gamma_a^d \gamma_b^e \gamma_c^f (0 + \nabla_d n_e n^g)(\nabla_g n_f) = -\gamma_a^d \gamma_b^g \gamma_c^f \nabla_d \nabla_g n_f - \gamma_a^d \gamma_b^e \gamma_c^f (\nabla_d n_e) n^g \nabla_g n_f = -\gamma_a^d \gamma_b^e \gamma_c^f \nabla_d \nabla_e n_f - \gamma_b^e \gamma_c^f (-K_{ae}) n^g \nabla_g n_f = -\gamma_a^d \gamma_b^e \gamma_c^f \nabla_d \nabla_e n_f - \gamma_c^f (-K_{ab}) n^g \nabla_g n_f$. Thus $D_a K_{bc} - D_b K_{ac} = -2 \gamma_a^d \gamma_b^e \gamma_c^f \nabla_{[d} \nabla_{e]} n_f - 0 = -\gamma_a^d \gamma_b^e \gamma_c^f R_{def}{}^g n_g$, which gives the Codazzi-Mainardi relation

$$ D_a K_{bc} - D_b K_{ac} = -\gamma_a^d \gamma_b^e \gamma_c^f n^g R_{defg}. $$

2. We start with $n^d \nabla_d n_a = n^d \nabla_d (-\alpha \nabla_a t) = -n^d (\nabla_d \alpha) \nabla_a t - n^d \alpha \nabla_a \nabla_d t = n_a n^d \nabla_d \ln \alpha + n^d \alpha \nabla_a (n_d / \alpha) = n_a n^d \nabla_d \ln \alpha - \alpha \nabla_a (1/\alpha) = n_a n^d \nabla_d \ln \alpha + \nabla_a \ln \alpha$ which we can rewrite as $n^d \nabla_d n_a = g_a^b (\nabla_b \ln \alpha + n^b n^d \nabla_d \ln \alpha) = (\gamma_a^b - n_a n^b)(\nabla_b \ln \alpha + n^b n^d \nabla_d \ln \alpha) = D_a \ln \alpha + 0 - n^a (n^b \nabla_b \ln \alpha - n^d \nabla_d \ln \alpha) = D_a \ln \alpha$. So

$$ n^d \nabla_d n_a = D_a \ln \alpha $$

Next we consider $K_{bc} = -\gamma_b^d \nabla_d n_c = -(g_b^d + n_b n^d) \nabla_d n_c = -\nabla_b n_c - n_b D_c \ln \alpha$, so

$$ \nabla_b n_c = -K_{bc} - n_b D_c \ln \alpha. $$

To make progress note $R_{abcd} n^d = \nabla_a \nabla_b n_c - \nabla_b \nabla_a n_c = -(\nabla_a (K_{bc} + n_b D_c \ln \alpha) - \nabla_b (K_{ac} + n_a D_c \ln \alpha)) = \nabla_b K_{ac} + (\nabla_b n_a) D_c \ln \alpha + n_a \nabla_b D_c \ln \alpha - (a \leftrightarrow b) = \nabla_b K_{ac} - (K_{ba} + n_b D_a \ln \alpha) D_c \ln \alpha + n_a \nabla_b D_c \ln \alpha - (a \leftrightarrow b)$

So $\gamma_e^a n^b \gamma_g^c (\nabla_a \nabla_b n_c - \nabla_b \nabla_a n_c) = \gamma_e^a n^b \gamma_g^c (\nabla_b K_{ac} - (K_{ba} + n_b D_a \ln \alpha) D_c \ln \alpha + n_a \nabla_b D_c \ln \alpha - (a \leftrightarrow b)) = \gamma_e^a n^b \gamma_g^c (\nabla_b K_{ac} - n_b (D_a \ln \alpha) D_c \ln \alpha - \nabla_a K_{bc} + n_a (D_b \ln \alpha) D_c \ln \alpha - n_b \nabla_a D_c \ln \alpha) = \gamma_e^a \gamma_g^c (n^b (\nabla_b K_{ac} - \nabla_a K_{bc}) + (D_a \ln \alpha) D_c \ln \alpha + \nabla_a D_c \ln \alpha) = \gamma_e^a \gamma_g^c (n^b \nabla_b K_{ac} + K_{bc} \nabla_a n^b + (D_a \ln \alpha) D_c \ln \alpha + \nabla_a D_c \ln \alpha)$, where in the last step we used $-n^b \nabla_a K_{bc} = K_{bc} \nabla_a n^b - \nabla_a (n^b K_{bc})$.

So now $\gamma_e^a n^b \gamma_g^c (\nabla_a \nabla_b n_c - \nabla_b \nabla_a n_c) = \gamma_e^a \gamma_g^c (n^b \nabla_b K_{ac} + K_c{}^b (-K_{ab} - n_a D_b \ln \alpha) + (D_a \ln \alpha) D_c \ln \alpha + \nabla_a D_c \ln \alpha) = \gamma_e^a \gamma_g^c (n^b \nabla_b K_{ac} - K_c{}^b K_{ab} + (D_a \ln \alpha) D_c \ln \alpha + \nabla_a D_c \ln \alpha)$

This already contains $n^b \nabla_b K_{ac}$ which is one term of a Lie derivative: $\pounds_n K_{ac} = n^b \nabla_b K_{ac} + K_{bc} \nabla_a n^b + K_{ab} \nabla_c n^b = n^b \nabla_b K_{ac} + K_c{}^b (-K_{ab} - n_a D_b \ln \alpha) + K_a{}^b (-K_{cb} - n_c D_b \ln \alpha)$

Since $\pounds_n K_{eg}$ is purely spatial we can write it as $\pounds_n K_{eg} = \gamma_e^a \gamma_g^c \pounds_n K_{ac} = \gamma_e^a \gamma_g^c (n^b \nabla_b K_{ac} - K_c{}^b K_{ab} - K_a{}^b K_{cb}) = \gamma_e^a \gamma_g^c (n^b \nabla_b K_{ac} - 2K_c{}^b K_{ab})$
With this $\gamma_e^a n^b \gamma_g^c (\nabla_a \nabla_b n_c - \nabla_b \nabla_a n_c) = \gamma_e^a \gamma_g^c (\pounds_n K_{ac} + 2K_c{}^b K_{ab} - K_c{}^b K_{ab} + (D_a \ln \alpha) D_c \ln \alpha + \nabla_a D_c \ln \alpha) = \pounds_n K_{eg} + K_g{}^b K_{eb} + (D_e \ln \alpha) D_g \ln \alpha + D_e D_g \ln \alpha$.
Now note $D_e D_g \ln \alpha = D_e((D_g \alpha)/\alpha) = (D_e D_g \alpha)/\alpha - (D_g \alpha)(D_e \alpha)/\alpha^2 = (D_e D_g \alpha)/\alpha - (D_e \ln \alpha) D_g \ln \alpha$.
Thus we finally get $\gamma_e^a n^b \gamma_g^c R_{abcd} n^d = \gamma_e^a n^b \gamma_g^c (\nabla_a \nabla_b n_c - \nabla_b \nabla_a n_c) = \pounds_n K_{eg} + K_g{}^b K_{eb} + (D_e D_g \alpha)/\alpha$
So finally:

$$\gamma_e^a n^b \gamma_g^c R_{abcd} n^d = \pounds_n K_{eg} + K_g{}^b K_{eb} + \frac{D_e D_g \alpha}{\alpha}$$

3. From $W = e^{-2\phi}$ and $\phi = \frac{1}{12} \ln \gamma$ we get

$$\partial_t W = \frac{W}{3}(\alpha K - \partial_i \beta^i) + \beta^i \partial_i W$$