Numerical Relativity - PHY 6938

Solutions to HW 3

1. If $G = c = 1$:
a) $1 = c \approx 3 \times 10^8$m/s, thus 1s$\approx 3 \times 10^8$m
$1 = G/c^2 \approx [6.7 \times 10^{-11}/(3 \times 10^8)^2]$m/kg, thus 1kg$\approx 7.4 \times 10^{-28}$m
$1\text{m/s} = 1\text{m}/(3 \times 10^8\text{m}) \approx 3 \times 10^{-9}$
$1\text{m/s}^2 = 1\text{m}/(9 \times 10^{16}\text{m*m}) \approx 1 \times 10^{-17}/\text{m}$
b) $m\omega = 0.05$ thus $\omega = 0.05/1000$m, but 1m$\approx 3 \times 10^{-9}$s. So $\omega \approx 15000/$s

2. a) Since $n_a = -\alpha \nabla_a t$ and $x^0 = t$ we obtain $n_\mu = (-\alpha, 0, 0, 0)$.
Consider $t^\mu = \frac{dx^\mu}{dt}$, then $t^\mu = (1, 0, 0, 0)$. With $\alpha n^a + \beta^a = t^a$ this leads to $n^\mu = (1, -\beta^i)/\alpha$
b) $\gamma_{ab} := g_{ab} + n_a n_b$. So since $\gamma^{ab} n_a = 0$ we have $\gamma^{\mu 0} = 0$ and $\gamma_{ij} = g_{ij}$. One can show that $g_{0k} = \beta_k$ and $g_{00} = -\alpha^2 + \beta_k \beta^k$, thus $\gamma_{0k} = \beta_k$ and $\gamma_{00} = g_{00} + n_0 n_0 = \beta_k \beta^k$
c) $\gamma^{ik}\gamma_{kj} = \gamma^{i\mu}\gamma_{\mu j} = (g^{i\mu} + n^i n^\mu)\gamma_{\mu j} = g^{i\mu}\gamma_{\mu j} = g^{i\mu}(g_{\mu j} + 0) = \delta^i_j$

3. $\gamma_{ab} := g_{ab} + n_a n_b$
a) Since $\gamma^c_a n^a = 0$, $K_{ab} n^a = 0 = K_{ab} n^b$
b) $K_{ab} := -\gamma^c_a \gamma^d_b \nabla_c n_d = -\gamma^c_a \gamma^d_b (\nabla_d t \nabla_c k + k \nabla_c \nabla_d t) = -\gamma^c_a \gamma^d_b ([n_d/k]\nabla_c k + k \nabla_c \nabla_d t) = -\gamma^c_a \gamma^d_b k \nabla_c \nabla_d t$. Since $\nabla_c \nabla_d t$ is symmetric, K_{ab} is also symmetric.
c) Since $n_d n^d = -1$ we have $n^d \nabla_c n_d = 0$. $K_{ab} = -\gamma^c_a (g^d_b \nabla_c n_d + n_b n^d \nabla_c n_d) = -\gamma^c_a \nabla_c n_b + 0$
d) $\pounds_n \gamma_{ab} = \pounds_n (g_{ab} + n_a n_b) = \nabla_a n_b + \nabla_b n_a + n^c \nabla_c (n_a n_b) + n_c n_b \nabla_a n^c + n_a n_c \nabla_b n^c = \nabla_a n_b + \nabla_b n_a + n_b n^c \nabla_c n_a + n_a n^c \nabla_c n_b + 0 = (g^c_a + n_a n^c)\nabla_c n_b + (g^c_b + n_b n^c)\nabla_c n_a = \gamma^c_a \nabla_c n_b + \gamma^c_b \nabla_c n_a = -K_{ab} - K_{ba} = -2K_{ab}$

4.
a) $\pounds_T S^a{}_b = T^c \nabla_c S^a{}_b + S^a{}_c \nabla_b (\alpha n^c) - S^c{}_b \nabla_c (\alpha n^a) = \alpha n^c \nabla_c S^a{}_b + \alpha S^a{}_c \nabla_b n^c - \alpha S^c{}_b \nabla_c n^a - S^c{}_b n^a \nabla_c \alpha$ since $S^a{}_b n_a = 0 = S^a{}_b n^b$. So $n^b \pounds_T S^a{}_b = \alpha n^b n^c \nabla_c S^a{}_b + \alpha S^a{}_c n^b \nabla_b n^c - 0 = \alpha(n^b n^c \nabla_c S^a{}_b + S^a{}_b n^c \nabla_c n^b) = \alpha[n^c \nabla_c (n^b S^a{}_b)] = 0$. We can also see that $(\nabla_a t)\pounds_T S^a{}_b = \pounds_T (S^a{}_b \nabla_a t) - S^a{}_b \pounds_T \nabla_a t$. Now we use $S^a{}_b \nabla_a t = 0$ and $\pounds_T \nabla_a t = \nabla_a \pounds_T t$ to obtain $(\nabla_a t)\pounds_T S^a{}_b = -S^a{}_b \nabla_a \pounds_T t$. Now use $\pounds_T t = T^c \nabla_c t = t^c \nabla_c t = \partial_t t = 1$ where we defined $t^c = T^c + \beta^c$. So finally $(\nabla_a t)\pounds_T S^a{}_b = -S^a{}_b \nabla_a 1 = 0$, and thus $(\pounds_T S^a{}_b) n_a = 0$
Thus we found $(\pounds_T S^a{}_b) n_a = 0 = (\pounds_T S^a{}_b) n^b$ and $\pounds_T S^a{}_b = \alpha \pounds_n S^a{}_b$.
b) Using the usual Lie drivative formula as in a) we see that $\pounds_T S^a{}_b = \alpha \pounds_n S^a{}_b + S^a{}_c n^c \nabla_b \alpha - S^c{}_b n^a \nabla_c \alpha = \alpha \pounds_n S^a{}_b - S^c{}_b n^a \nabla_c \alpha$. We thus find $(\pounds_n S^a{}_b) n^b = 0$.
But $(\alpha \pounds_n S^a{}_b) n_a = n_a \pounds_T S^a{}_b - 0 + S^c{}_b n^a \nabla_c \alpha = 0 + S^c{}_b n^a \nabla_c \alpha$. The latter is not necessarily zero. It is zero only if the lapse is constant over spatial slices.

Note: $(\pounds_n S^a{}_b) n_a = \pounds_n (S^a{}_b n_a) - S^a{}_b \pounds_n n_a$ and $\pounds_n n_a = n^c \nabla_c n_a + n_c \nabla_a n^c = n^c \nabla_c n_a + 0$. Furthermore $n^c \nabla_c n_a = n^c \nabla_c (-\alpha \nabla_a t) = -\alpha n^c \nabla_c \nabla_a t - (\nabla_a t) n^c \nabla_c \alpha = -\alpha n^c \nabla_a \nabla_c t - (\nabla_a t) n^c \nabla_c \alpha = \alpha n^c \nabla_a (n_c/\alpha) + (n_a/\alpha) n^c \nabla_c \alpha = \alpha[(n_c/\alpha)\nabla_a n_c - n^c n_c (\nabla_a \alpha)/\alpha^2 + (n_a/\alpha) n^c \nabla_c \alpha = 0 + (\nabla_a \alpha)/\alpha + (n_a/\alpha) n^c \nabla_c \alpha = \nabla_a \ln \alpha + n_a n^c \nabla_c \ln \alpha$. Which gives the same result.