Numerical Relativity - PHY 6938

HW 9

Hand in this homework.

READ: Chap 7.

PROBLEMS:

1. The first law of thermodynamics is usually written as

$$dE = TdS - pdV + \mu dN$$

Here E is the total relativistic energy.
a) Define $\rho := E/V$, $n := N/V$, $s := S/N = S/(Vn)$. Calculate $d\rho$ in terms of ds and dn.
b) Calculate $d\rho$ in terms of ds and $d\rho_0$, where ρ_0 is the rest mass density.

2. Consider a polytropic EoS

$$p = \kappa \rho_0^{1+1/n}$$

at $T = 0$. Note here n is the polytropic index.
a) Using this EoS express ρ_0, ϵ and p in terms of h, n and κ. (Recall that h is given by $h = 1 + \epsilon + p/\rho_0$.)
HINT: Use $T = 0$ in the expression for dh to obtain an equation that you can integrate to find h as a function of ρ_0.
b) Express ρ_0, ϵ and h in terms of p, n and κ.

3. Let us now consider a static spherically symmetric star. In this case the Einstein equations can be solved exactly. The metric outside the star is given by the Schwarzschild metric

$$ds^2 = -(1 - 2m(r_*)/r)dt^2 + (1 - 2m(r_*)/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

where r_* is the star radius (in standard Schwarzschild coordinates) and

$$m(r) = \int_0^r dr' 4\pi r'^2 \rho.$$

Inside the star the metric is given by

$$ds^2 = -e^{2\phi}dt^2 + (1 - 2m(r)/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

where $m(r)$, $\phi(r)$ and also $p(r)$ are found by integrating the Tolman-Oppenheimer-Volkoff (TOV) equations

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{1}$$

$$\frac{dp}{dr} = -(\rho + p)(m + 4\pi r^3 p)/(r(r - 2m)) \tag{2}$$

$$\frac{d\phi}{dr} = (m + 4\pi r^3 p)/(r(r - 2m)). \tag{3}$$

The TOV equations are ordinary differential equations that have to be integrated out from $r = 0$ to the point where $p = 0$, which is the star surface location ($r = r_*$). At $r = 0$ we start with $m = 0$ and some value of $p = p_c$ which determines the core pressure and thus the total mass of the star. We also have to start with some particular ϕ at $r = 0$. We can set this value to 1 at first. The final $\phi(r)$ can be obtained by adding a constant to it such that $2\phi(r_*) = \ln(1 - 2m(r_*)/r_*)$. This shift in ϕ ensures that the metric is continuous at the star surface.

a) Assume that the equation of state is

$$p = \kappa \rho_0^{1+1/n}$$

and that $T = 0$. Express ρ_0 and ρ in terms of p.

b) Convince yourself that $m(r) \approx \frac{4\pi}{3} r^3 \rho$ for small r. Write the TOV equations for small r.

c) Write a program in Python (or optionally in C) that integrates the TOV equations. Use units such that $G = c = M_\odot = 1$. To do so notice that for $r = 0$ the full TOV equations contain terms that blow up. Use your result from b) for small r and the full TOV equations for all other r. In order to get results quickly, use the simple Euler method for the numerical integration (this can always be replaced by a Runge-Kutta integrator later if you get ambitious).

Recall: The Euler method for an equation of the form $\partial_r \vec{u} = \vec{f}(\vec{u}, r)$ simply is $\vec{u}(r_{n+1}) = \vec{u}(r_n) + \vec{f}(\vec{u}(r_n), r_n)\Delta r$, where $r_n = n\Delta r$. An example can be found in `ODE_Euler.py` where the variable t is used instead of r.

d) Consider the EoS with $\kappa = 123.6489$ and $n = 1$ (in units such that $G = c = M_\odot = 1$). Run your program with $p_c = 8 \times 10^{-4}$ and attach a plot of $p(r)$.

[The command

`tgraph.py -c 1:3 f.dat`

will plot column 3 vs column 1 in file `f.dat`]

e) Run your program with different p_c to approximately find the maximum $m(r_*)$ that is possible for the EoS with $\kappa = 123.6489$ and $n = 1$ (in units such that $G = c = M_\odot = 1$). Note that stars with p_c beyond the maximum are unstable.