Numerical Relativity - PHY 6938

HW 6

Hand in problems 2. and 3. of this homework.

READ: Chap 4., Appendix B.

NOTE: This HW is about the generalized harmonic system that is often used in numerical evolutions instead of the 3+1 based BNSSOK or Z4 systems.
Since 1. is a bit tedious, you are not required to hand it in.

PROBLEMS:

1. Show that the Ricci tensor can be written as

$$R_{ab} = -\frac{1}{2} g^{cd}\partial_c\partial_d g_{ab} + \nabla_{(a}\Gamma_{b)} + g^{cd}g^{ef}\left(\partial_e g_{ca}\partial_f g_{db} - \Gamma_{ace}\Gamma_{bdf}\right), \tag{1}$$

in any coordinate system. Here

$$\nabla_a\Gamma_b := \partial_a\Gamma_b - g^{cd}\Gamma_{cab}\Gamma_d,$$

where ∇_c denotes the covariant derivative compatible with g_{ab}, and $\Gamma_a \equiv g^{bc}\Gamma_{abc}$ is the trace of the standard Christoffel symbol Γ_{abc}:

$$\Gamma_{abc} = \frac{1}{2}(\partial_b g_{ac} + \partial_c g_{ab} - \partial_a g_{bc}). \tag{2}$$

NOTE:
In harmonic coordinates, $\Gamma_a = 0$, so the only second-derivative term remaining in the Ricci tensor is $g^{cd}\partial_c\partial_d g_{ab}$. Therefore in harmonic coordinates the vacuum Einstein equations, $R_{ab} = 0$, form a manifestly hyperbolic system.

2. Consider $\Box x^\alpha = g^{\mu\nu}\nabla_\mu\nabla_\nu x^\alpha$.
a) Express $\Box x^\alpha$ in terms of partial derivatives of $\mathfrak{g}^{\mu\alpha} := \sqrt{|g|}g^{\mu\alpha}$
b) Express $\Box x^\alpha$ in terms of Γ^α
c) Express $\Box x^0 = \Box t$ in terms of lapse derivatives and K. HINT: start from $\nabla^\mu n_\mu$.
d) Relate $\Box x^i$ to derivatives of the shift (and lapse) and Christoffel symbols.
HINT: expressing $\Box x^i = (\gamma^{\mu\nu} - n^\mu n^\nu)\nabla_\mu\nabla_\nu x^i$ and noting that $n^\nu\nabla_\nu x^i$ is a scalar (that can be expressed in terms of lapse and shift) may be helpful.

3. Generalized Harmonic formulation:
Choose coordinates such that

$$g_{\alpha\beta}\Box x^\beta = H_\alpha,$$

where $H_\alpha = H_\alpha(t, x^i, g_{\mu\nu})$ are functions that can be freely chosen.
a) Use your results from 1. & 2. to find both $(\partial_t - \beta^k\partial_k)\alpha$ and $(\partial_t - \beta^k\partial_k)\beta^i$ in terms of H_α.

b) Rewrite the Einstein equations (of the form $R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$) in terms of $g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta}$ and first derivatives of $g_{\mu\nu}$ and H_α.
c) Compare your result from b) to a wave equation $\Box\phi = s$ with source s. Argue that the principal terms (the terms with the highest derivatives) obey wave equations. [This is important, because it means the system is symmetric hyperbolic and thus well-posed with nice numerical properties.]