Numerical Relativity - PHY 6938

HW 10

Hand in this homework.

READ: Chap 7, 9.

PROBLEMS:

1. Consider an advection equation of the form

$$\partial_t u + v\partial_x u = 0$$

In homework 8 we have seen that the discretization

$$u_m^{n+1} = u_m^n - \frac{v\Delta t}{2\Delta x}(u_{m+1}^n - u_{m-1}^n)$$

is unstable. Replace u_m^n on the right hand side by $\frac{1}{2}(u_{m+1}^n + u_{m-1}^n)$ to obtain the Lax method

$$u_m^{n+1} = \frac{1}{2}(u_{m+1}^n + u_{m-1}^n) - \frac{v\Delta t}{2\Delta x}(u_{m+1}^n - u_{m-1}^n).$$

a) Find the order of accuracy of the Lax method in both space and time.
b) Perform a von Neumann stability analysis and compute $|\xi|$. Under what conditions is the Lax method stable?

2. A toy Riemann problem:
Consider Burgers' equation

$$\partial_t u + \partial_x f(u) = 0$$

with $f(u) = u^2/2$.
In class we have seen that a discontinuity

$$u(0,x) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases}$$

has two types of solutions. If $u_l > u_r$ we get a shock wave of the form

$$u(t,x) = \begin{cases} u_l & \text{if } x < st \\ u_r & \text{if } x > st \end{cases},$$

where $s = \frac{f(u_r)-f(u_l)}{u_r-u_l}$. If $u_l < u_r$ we get a rarefaction wave of the form

$$u(t,x) = \begin{cases} u_l & \text{if } x < u_l t \\ x/t & \text{if } u_l t < x < u_r t \\ u_r & \text{if } x > u_r t \end{cases}.$$

a) Define $\tilde{u}(t) := u(t,0)$. Find $\tilde{u}(t)$ if $u_l > 0$ and $u_r > 0$.

b) Find $\tilde{u}(t)$ if $u_l < 0$ and $u_r < 0$.
c) Find $\tilde{u}(t)$ if $u_l > 0$ and $u_r < 0$.
d) Find $\tilde{u}(t)$ if $u_l < 0$ and $u_r > 0$.
e) Find the flux

$$F^{n+1/2}(x=0) = \frac{1}{\Delta t}\int_0^{\Delta t} dt f(\tilde{u}(t+n\Delta t))$$

averaged over a time interval Δt at $x = 0$ for all signs of u_l and u_r.

3. Consider again Burgers' equation

$$\partial_t u + u\partial_x u = 0.$$

a) Consider initial data of the form

$$u(0,x) = \begin{cases} 1.5 & \text{if } x < 1 \\ 0.5 & \text{if } x \geq 1 \end{cases} .$$

Study and run `Burgers1.py` which is using standard finite differences instead of fluxes (use e.g. `tgraph.py` to see how u evolves). What discretization are we using? Approximately where is the discontinuity at $t = 14$? Where should it be at $t = 14$?
b) Look at `Burgers_f.py` which uses fluxes. It uses the same grid as `Burgers1.py`. Each grid point $x = x_m$ is at the center of a cell. The cell boundaries $x_{m+1/2}$ are halfway between grid points. One uses fluxes at $x_{m+1/2}$ and $x_{m-1/2}$ to determine how u changes at x_m. Note that `Burgers_f.py` does not work because `F_interface(ul, ur)` uses the wrong approximation for the flux at the interface between two cells!
Modify `Burgers_f.py` such that it uses the flux $F^{n+1/2}(x = 0)$ you found in problem 2, i.e. solve a Riemann problem at each cell interface and assume that u is constant and equal to u_m within cell m (as in Godunov's method). Run your improved program. Approximately where is the discontinuity at $t = 14$?
(Optionally you can also write your own program in C.)
c) Attach a printout of your improved program and a plot of $u(t = 14, x)$ for both `Burgers1.py` and your improved program.
d) Which program is best able to evolve the shock wave?